THE MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

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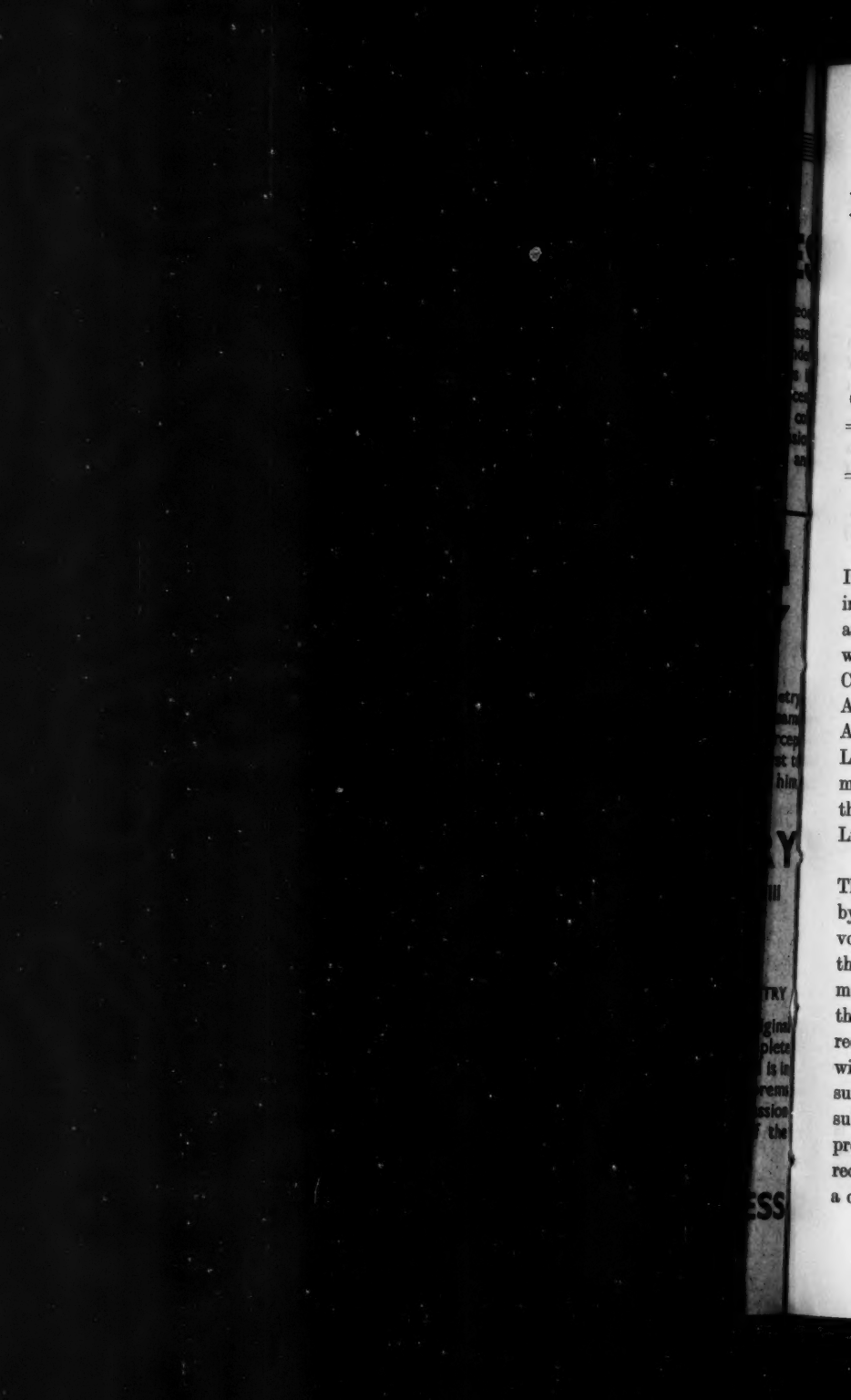

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VOL. XXIII. OCTOBER, 1939. No. 256

EDITORIAL NOTE.

IT is clearly impossible at this stage to offer any very definite indications of the ways in which the activities of the Association are likely to be affected by the war. Steps are being taken to deal with matters of urgency, and it is possible that an Executive Committee may be constituted with power to act for the Council. Although no decisions have yet been taken, it seems certain that the Annual Meeting of the Association will have to be abandoned. The Librarian hopes to be able to keep the Library at the service of members, but abnormal conditions demand certain modifications in the regulations; these modifications are set out in a note on the Library printed in the Inset to this number of the *Gazette*.

Publication of the *Gazette* will continue, at any rate for the present. The Editor hopes to complete the present volume in normal fashion by publication of a December number; but plans for the next volume are necessarily vague. An annual volume of three numbers, themselves smaller than the recent average, may be possible. But members will understand that the governing factor is likely to be the income of the Association, and this will no doubt be drastically reduced. The question of subscriptions from members on service will be considered by the Council or by its Executive Committee. If sufficient members of the Association are able to continue their subscriptions in spite of war-time burdens, then there is every prospect of the *Gazette* continuing publication, though on a much reduced scale, and so avoiding the difficulties of a resumption after a complete break.

APPLICATIONS OF ELEMENTARY MATHEMATICS.

Answers to the appeal, *Gazette*, October, 1938.

I. *Angle properties of circle. Setting out a circular arc.*

The problem is to locate a series of points on a circular arc, in railway or canal construction. The end points, T_1 and T_2, and the angle α at which the tangents at these points intersect, are known. If R is any point on the arc and Q is a point on T_1R produced, then $\angle QRT_2$ is equal to $\alpha/2$, so that a mirror in a vertical plane at R with its normal at an angle $\alpha/4$ to RQ will reflect rays coming from T_2 in the direction RQ. This is the principle underlying the construction of a new instrument. The mirror is fixed at the appropriate angle to a line of sights. This line of sights is directed to a range-pole at T_1, and the instrument moved until the image of another range-pole at T_2 appears to be in the direction RT_1.

(Sent by F. F. Ferguson, P.W.D., Jodhpur, Rajputana, India. Cf. *Amer. Soc. C.E.*, " Civil Engineering ", 8, pp. 548-9, Aug., 1938.)

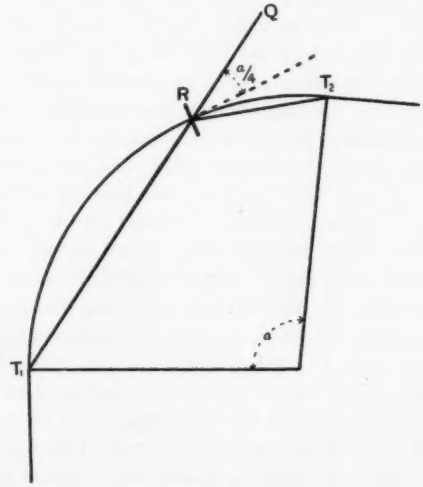


FIG. 1.

II. *Properties of parallelogram, applied in various mechanisms.*

(1) Locomotive coupling rod.

Since the length of the coupling rod (between bearing centres) is equal to that between the axles, and the cranks are of equal length, the cranks must always be parallel, and must therefore turn at equal speeds. The same principle is applied in

(2) Coupled switches (see Fig. 2).

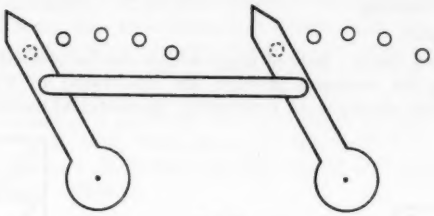


FIG. 2.

(3) Dual wind-screen wipers.

(4) Roberval's balance (see Fig. 3). The supports of the pans are coupled by equal arms pivoted to the case at their mid-points, so that the supports remain vertical, and the pans horizontal, in all positions.

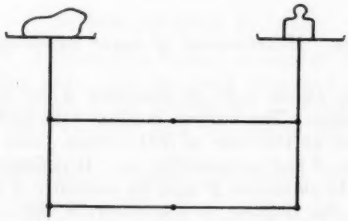


FIG. 3.

(5) Link support for front wheels of some cars, and motor cycles (see Fig. 4). The wheel can surmount a bump without altering the height or inclination of the body of the vehicle.

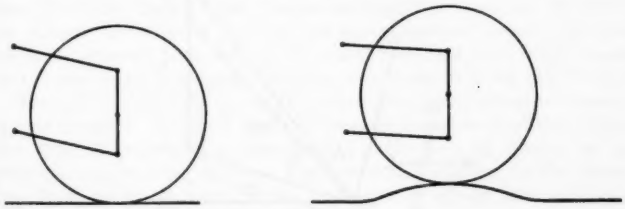


FIG. 4.

(6) "Lazy-tongs." These involve the expanding and contracting possibilities of a deformable parallelogram. In many applications of the principle, the fact that the diagonals of a rhombus are perpendicular is also implicit. Instances are :

(7) Wall brackets for telephones, etc.

(8) Trellis fencing.

(9) Lift gates.

(10) Folding trays. Sets of trays which can be opened for display and fold up for compact storage are illustrated in Fig. 5. The location of the pivots is an interesting geometrical exercise.

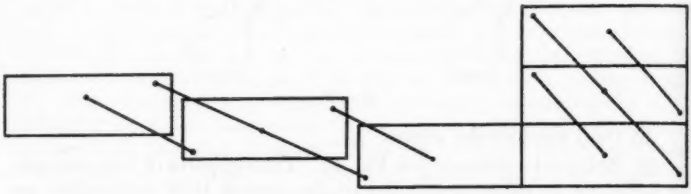


FIG. 5.

(Compiled from notes supplied by S. Inman, Isleworth County School.)

III. *Trigonometry. Measurement of upper winds by means of pilot balloons.*

Small balloons, about 2 ft. in diameter when full, are released into the atmosphere. The balloon is filled with sufficient hydrogen to cause it to rise at the rate of 500 ft./min., and it takes up the horizontal motion of the surrounding air. It is followed by a special theodolite, and its elevation E and its azimuth A are noted every minute. If B is the balloon, O the observer, $BC=h$ the height of the balloon, and $OC=D$, then $D=h\cot E$ and the components of D are

$$D_N=h\cot E\cos A,\quad D_E=h\cot E\sin A.$$

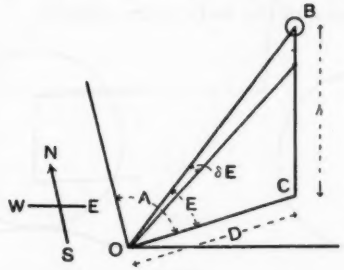


FIG. 6.

From successive values of these components the wind speed and direction at the average height of the balloon during the minute is easily obtained. A special slide rule has been designed to facilitate the computation.

It is found that the assumption of a constant rate of rise is usually sufficiently accurate. If greater accuracy is desired, simultaneous observation from two theodolites are taken, and the reduction of the observations is then considerably more complicated. A middle course is to suspend a piece of silver paper from the balloon, and to obtain the difference between the elevation of this and of the balloon by means of a graticule incorporated in the telescope.

(Sent by G. A. Bull, Meteorological Office, R.A.F., Felixstowe. Further details in a pamphlet with the above title obtainable from H.M. Stationery Office.)

It is hoped to publish further replies in due course. Additional contributions should be sent to A. Buxton, 37 Heddon, Whitchurch, Cardiff.

JOHN JAMES MILNE

THE Reverend J. J. Milne was elected a member of the Association on January 22nd, 1884; by his death this year we have lost one of our oldest members. He served the Association as Honorary Treasurer from 1897 to 1899, and was one of the Honorary Secretaries in 1896.

Milne's *Weekly Problem Papers*, *Companion* and *Solutions*, were well known to coaches; with R. F. Davis, he produced a *Geometrical Conics*. His most important book was doubtless the *Elementary treatise on cross-ratio geometry*, notable for its concise power and elegance, and for its historical notes, many of them intended to show how the roots of cross-ratio geometry are to be found in the Greek geometry. Milne's love of Greek geometry led him to a discovery which gave him keen pleasure: Apollonius had stated that the third book of his *Conics* contained a complete solution of the famous problem of the *locus ad tres et quatuor lineas*, but it was taken for granted that this solution had been lost until Milne showed that it was given in a somewhat disguised form in Book III, Prop. 54.

Many of Milne's gifts to the Library of the Association are recorded in past *Gazettes*. He took great interest in the growth of the Library, and that it possesses a remarkable collection of books on geometrical conics, ranging from the sixteenth century to the present day, is in no small part due to a generosity which was one aspect of Milne's devotion to his chosen subject.

MUTUALITY.*

By N. M. Gibbins.

1. This lecture grew out of an attempt to trace the consequences of putting together two examination questions. The subject-matter of both is the reflections X, Y, Z of a point P in the sides of a triangle ABC. In the first question we have to show that the perpendiculars from A on YZ, from B on ZX, from C on XY meet in P', the centre of the circle XYZ, and that the relation between P and P' is mutual.

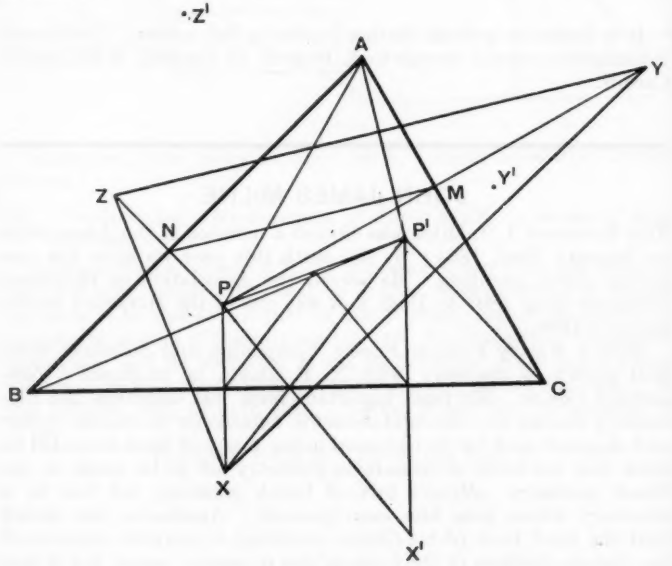


Fig. 1.

Since $AY=AP=AZ$, the perpendicular from A to YZ bisects it. Hence this perpendicular passes through the centre of the circle XYZ, as do similarly the other two perpendiculars. Let X', Y', Z' be the reflections of P' in the sides of ABC. Since BC is the common perpendicular bisector of PX and $P'X'$, $PX'=P'X$, and similarly $PY'=P'Y$, $PZ'=P'Z$. Hence P is the centre of the circle $X'Y'Z'$, and the two radii are equal. Also by proportionate division the six feet of the perpendiculars from P and P' to the sides all lie on a circle whose centre is the middle point of PP', and whose radius is half that of the other two.

* A lecture given to the Oxford University Invariant Society on 17th February, 1939.

This last property is characteristic of isogonal conjugates, and from Fig. 1 $AMPN$ is a cyclic quadrilateral so that $AMN = APN$. Also MN is parallel to YZ so that AP' is perpendicular to MN. Hence the complements $P'AC$, PAB are equal. Similarly $P'BC = PBA$ and $P'CA = PCB$. Hence if X, Y, Z are the reflections of P in the sides of ABC, the centre of the circle XYZ is the isogonal conjugate of P with respect to ABC, where now isogonal conjugates are defined in the usual way.

2. The second question is to prove that in Fig. 1 the circles BXC, CYA, AYB have a common point Q, and that in general the relation between P and Q is mutual.

Let the circles BXC, AZB meet in Q. Then

$$BQC = 180° - BXC = 180° - BPC,$$

and

$$AQB = 180° - AZB = 180° - APB.$$

Hence $AQC = APC = AYC$, so that the circle CYA passes through Q. Let X_1 be the reflection of Q in BC. Then

$$BX_1C = BQC = 180° - BPC,$$

as above. Hence the circle BX_1C goes through P, as do similarly the circles CY_1A and AZ_1B, where Y_1 and Z_1 are the reflections of Q in CA and AB.

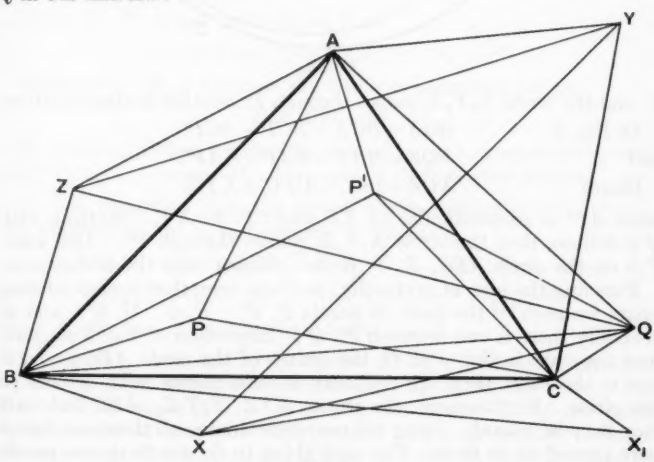


FIG. 2.

When P is on the circle ABC, the three circles pass through the orthocentre of ABC; and when P is the orthocentre, the three circles coincide with the circle ABC, so that now Q is any point thereon.

By § 1 the centre of the circle $X_1Y_1Z_1$ is Q', the isogonal conjugate

of Q with respect to the triangle ABC, and in the preceding paragraph the orthocentre and circumcentre are isogonal conjugates.

3. An accurately drawn figure suggests a number of further properties. It appears, for example, that the circle XYZ passes through

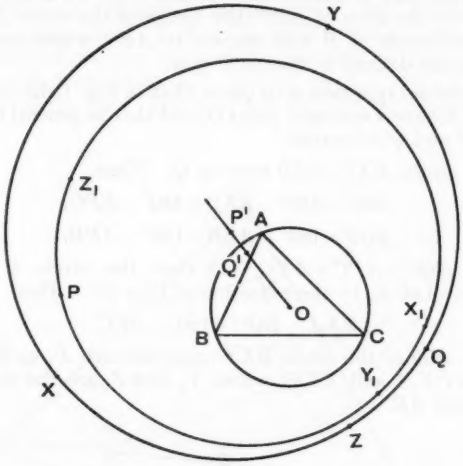


FIG. 3.

Q, and the circle $X_1Y_1Z_1$ passes through P, another mutual relation.

In Fig. 2 $\quad BQX = BCX = BCP = ACP'$

and $\quad BQZ = BAZ = BAP = CAP'.$

Hence $\quad XQZ = 180° - A'PC = XYZ,$

since AP' is perpendicular to YZ and CP' to XY. Starting with Q it follows that the circle $X_1Y_1Z_1$ passes through P'. Also when P is on the circle ABC, X, Y, Z are collinear with the orthocentre.

Pursuing the idea of mutuality, we have seen that such a relation exists for each of the pairs of points P, P'; P, Q; Q, Q'; and so probably there is one between P', Q'. Inspection of Fig. 3 suggests that they are in line with O, the centre of the circle ABC; and if this is the case, they are probably inverse points with respect to this circle. Furthermore, the circles XYZ, $X_1Y_1Z_1$, ABC look as if they may be coaxal. Using trilinear coordinates all these suspicions were proved to be facts. The next thing to do was to devise proofs in the spirit of pure geometry. The chain of reasoning is a long one and the links in it are as follows. The proofs are given in § 7.

4. (*a*) If L, M, N are the feet of the perpendiculars from P to the sides of a triangle ABC, and p the power of P with respect to the circle ABC, then the area of the triangle LMN is

$$\tfrac{1}{2}p \sin A \sin B \sin C.$$

Hence if P is (α, β, γ), $|p| = \Sigma\beta\gamma \sin A/\Pi \sin A$.

(*b*) For isogonal conjugates $P(\alpha, \beta, \gamma)$ and $P'(\alpha', \beta', \gamma')$,

$$\alpha\alpha' = \beta\beta' = \gamma\gamma' = pp'/4R^2,$$

and

$$|p'| = 8R^3\alpha\beta\gamma/p^2.$$

(*c*) The square of the radius of the circle XYZ, which is four times that of LMN, is $|p'| \Pi \sin A/\Pi \sin BPC$. Since BQC, CQA, AQB are either equal to or the supplements of BPC, CPA, APB, the squares of the radii of the circles XYZ, $X_1Y_1Z_1$ are as the powers of their centres.

(*d*) The power of a point on one circle with respect to another varies as its perpendicular distance from their radical axis. BC is the radical axis of $BXCQ$ and ABC.

Hence if Q is $(\alpha_1, \beta_1, \gamma_1)$, taking powers with respect to ABC,

$$(\text{power of } Q) : (\text{power of } X) = \alpha_1 : -\alpha,$$

whence

$$p_1/\alpha_1 + p/\alpha + 4R\cos A = 0,$$

with two similar equations, and so by (*b*)

$$\alpha_1'/p_1' + \alpha'/p' + \cos A/R = 0,$$

with two similar equations.

Hence Q', P', O are collinear, so that

$$(\alpha_1' - R\cos A)/(\alpha - R\cos A) = OQ'/OP'.$$

Eliminating $\cos A$ between the last two equations we get

$$OQ' \,.\, OP' = R^2.$$

(*e*) Taking this last result with that of (*c*), the squares of the radii of XYZ, $X_1Y_1Z_1$ are as the distances of their centres from O.

(*f*) The condition that the circles whose centres are O_1, O_2, O_3, and whose radii are r_1, r_2, r_3 should be coaxal is

$$r_1^2 \,.\, O_2O_3 + r_2^2 \,.\, O_3O_1 + r_3^2 \,.\, O_1O_2 + O_2O_3 \,.\, O_3O_1 \,.\, O_1O_2 = 0,$$

or,

$$O_2O_3(r_1^2 - O_1O_2 \,.\, O_1O_3) + (r_3^2 \,.\, O_1O_2 - r_2^2 \,.\, O_1O_3) = 0,$$

which in our case is satisfied by (*d*) and (*e*).

5. Referring back to § 1, the circles $BX'C$, $CY'A$, $AZ'B$ meet in a point Q_2 (say) with its reflections X_2, Y_2, Z_2 in the sides. The centre of the circle $X'Y'Z'$ is P and the centre Q_2' of the circle $X_2Y_2Z_2$ is the inverse of P with respect to the circle ABC. Hence the centres of the four circles XYZ, $X'Y'Z'$, $X_1Y_1Z_1$, $X_2Y_2Z_2$ lie on a circle orthogonal to the circle ABC.

We may also compare the radii of the circles $X_1Y_1Z_1$, $X_2Y_2Z_2$. We have

$$r_1^2/r^2 = OQ_1'/OP' = R^2/OP'^2,$$

$$r_2^2/r'^2 = OQ_2'/OP = R^2/OP^2, \quad r = r';$$

hence

$$r_1 : r_2 = OP : OP'.$$

For the Brocard points $O\Omega = O\Omega'$, so if we start with P at Ω, the radii of the two "Q-circles" are equal and their centres are equidistant from O. It may be verified that each Q-circle passes through the centre of the other, and that the two common radical axes are $K\Omega$ and $K\Omega'$, where K is the symmedian point of ABC.

6. The facts that $P'Q'$ passes through O and that P' and Q' are the isogonal conjugates of P and Q, suggest another property. Let the equation of $P'Q'$ be $l\alpha + m\beta + n\gamma = 0$, with

$$l\cos A + m\cos B + n\cos C = 0.$$

Then P and Q lie on the conic $l/\alpha + m/\beta + n/\gamma = 0$ passing through H, the orthocentre of ABC. Hence this conic is a rectangular hyperbola, and the idea of mutuality suggests that P and Q are opposite ends of a diameter. In Fig. 2 construct the rectangular hyperbola whose centre is the middle point of PQ and which passes through P and B. Then Q is the other end of the diameter through P, while BC and AB subtend supplementary angles at P and Q. Hence * C and A are also on the rectangular hyperbola, and further so is H. The centre of any rectangular hyperbola through A, B, C lies on the nine-points circle of ABC, and therefore the middle point of PQ is on it.

Let x, y, z be the reflections of P in AH, BH, CH and x_1, y_1, z_1 those of Q. Then AH subtends equal or supplementary angles at P and Q, so that the circle AHx passes through Q, and the circle AHx_1 passes through P. Similarly the circles BHy, CHz pass through Q and the circles BHy_1, CHz_1 through P. Thus Q is the "Q-point" of P with respect to each of the triangles BHC, CHA, AHB as well as with respect to ABC.

Hence the circles Xyz, Yzx, Zxy all pass through Q, and the circles Xyz, Yzx, Zxy all pass through P.

There are corresponding sets of concurrent circles for P' and Q_2, while the relations between all these new circles must be a subject of investigation of bewildering complexity not now to be pursued further.

7. Proofs of the statements made in § 4.

(a) In Fig. 4 (a), $LMN = LPN - MNP - MLP$

$$= 180° - B - MAP - MCP$$
$$= 180° - AA'C - MAP - MCA' + PCA'$$
$$= PCA'.$$

In Fig. 4 (b), $LMN = PMN - PML$

$$= MPL + PLM - MPN - MNP$$
$$= B + PCM - MBP$$
$$= AA'C + PCM - AA'C - ACA'$$
$$= PCA'.$$

* *Vide* Macaulay's *Geometrical Conics* (Cambridge Press), 1921, p. 157.

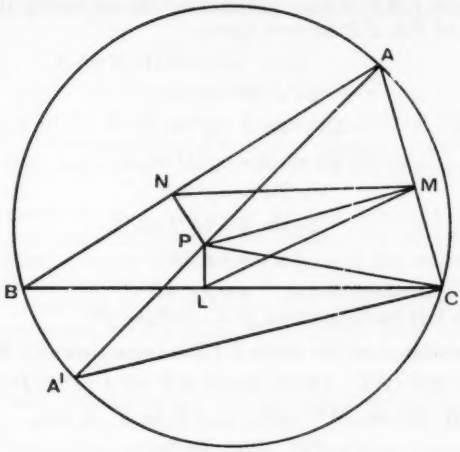


FIG. 4 (*a*).

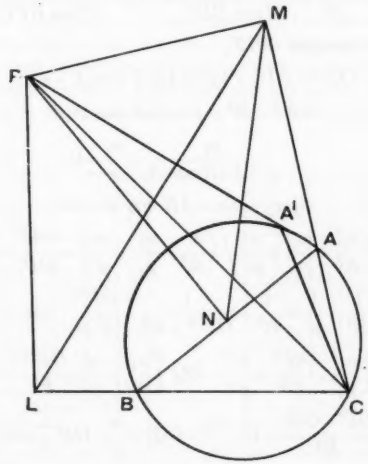


FIG. 4 (*b*).

In both figures,

$LM = PC \sin C$, $MN = PA \sin A$, $\sin ACA'/PA' = \sin B/PC$.

Hence $\frac{1}{2}LM \,.\, MN \sin LMN = \frac{1}{2}PA \,.\, PA' \sin A \sin B \sin C$

$= \frac{1}{2}(OP^2 \sim R^2)\, \Pi \sin A$.*

* Adapted from Coolidge's *Circle and Sphere* (Clarendon Press), 1916, p. 48.

The area of LMN is also $\frac{1}{2}\Sigma\beta\gamma \sin A$. Hence noting the signs of α, β, γ and of PA, PA' in each figure,

$$p = -(\Sigma\beta\gamma \sin A)/\Pi \sin A.$$

(*b*) Let $\alpha\alpha' = \beta\beta' = \gamma\gamma' \equiv \kappa.$

We have $-\Sigma\beta'\gamma' \sin A = p' \sin A,$

and so $-\alpha'\beta'\gamma'\Sigma\alpha \sin A/\kappa = p'\Pi \sin A,$

or $-\alpha'\beta'\gamma'/\kappa = p'/2R,$

since $\Sigma\alpha \sin A = 2R\Pi \sin A.$

Similarly $-\alpha\beta\gamma/\kappa = p/2R.$

Whence by multiplication, $\kappa = pp'/4R^2.$

From the last two equations $p' = -8R^3\alpha\beta\gamma/p^2.$

(*c*) The radius, r, of the circle XYZ is twice that of LMN, and so

$$r = MN \,.\, NL \,.\, LM/2 \,.\, \text{area } LMN = PA \,.\, PB \,.\, PC/p.$$

Now $PB \,.\, PC \sin BPC = BC \,.\, \alpha = 2R \sin A \,.\, \alpha,$ etc.

Hence $$r^2 = PA^2 \,.\, PB^2 \,.\, PC^2/p^2$$

$$= \frac{8R^3\alpha\beta\gamma}{p^2} \cdot \frac{\Pi \sin A}{\Pi \sin BPC} = -p' \cdot \frac{\Pi \sin A}{\Pi \sin BPC}.$$

(*d*) From the triangle OPX,

$$OX^2 = OP^2 + 4\alpha^2 + 4\alpha(R \cos A - \alpha).$$

Hence $$OX^2 - R^2 = p + 4\alpha R \cos A,$$

$$\frac{p_1}{p + 4\alpha R \cos A} + \frac{\alpha_1}{\alpha} = 0,$$

or $$p_1/\alpha_1 + p/\alpha + 4R \cos A = 0.$$

Also $$\left(\frac{\alpha_1'}{R^2} + \frac{\alpha_1'}{p_1'} + \frac{\alpha}{p}\right) \Big/ \left(\frac{\alpha'}{R^2} + \frac{\alpha_1'}{p_1'} + \frac{\alpha'}{p'}\right) = \frac{OQ'}{OP'}.$$

Since $$\frac{1}{R^2} + \frac{1}{p'} = \frac{1}{R^2} + \frac{1}{OP'^2 - R^2} = \frac{OP'^2}{R^2 p'}, \text{ etc.},$$

$$OP'\left[\frac{\alpha_1'}{p_1'} \cdot \frac{OQ'^2}{R^2} + \frac{\alpha}{p}\right] - OQ'\left[\frac{\alpha_1'}{p_1'} + \frac{\alpha'}{p'} \cdot \frac{OP'^2}{R^2}\right] = 0,$$

or $$\left(\frac{OP' \,.\, OQ'}{R^2} - 1\right)\left(\frac{\alpha_1'}{p_1'} \cdot OQ' - \frac{\alpha'}{p'} \cdot OP'\right) = 0,$$

with two similar equations. The vanishing of the second factor in each case would involve the coincidence of P' and Q'. Hence

$$OP' \,.\, OQ' = R^2.$$

(*e*) In Fig. 3,

$$(OQ'^2 - R^2)/(R^2 - OP'^2) = (OQ'^2 - OP' \,.\, OQ')/(OP' \,.\, OQ' - OP'^2)$$

$$= OQ'/OP'. \qquad \text{N. M. G.}$$

RULES FOR BORDERED MAGIC SQUARES.

By J. Travers.

I. Odd-Bordered Squares.

Rouse Ball in his *Mathematical Recreations* tells us that there are no rules for the construction of bordered squares; I have made the following rules, which are now published for the first time as far as I am aware.

It is remarkable that the rules for *odd* bordered squares are apparently not known as they are so simple. We use two diagrams, one a skeleton to make the method clear, and the other completed to show the full square.

Rule I. Place the letter M at the $\frac{1}{2}(n+1)$th square as shown in Fig. 1, where n denotes the order of the square (here $n=9$).

Rule II. Now, commencing at the second column, write the numbers 1, 2, 3, ... up to and under M. Then write the next number in the diagonal at the bottom right-hand corner; the next number in the same column and in the second row, and continue until we arrive at the square above M. Then write the next number at the left-hand side in row with M, and then the next numbers in the last row beginning at the column which is one square to the right of M, and continue until the spaces in the row are filled. Then put the next number in the diagonal at the bottom left-hand corner, and write upwards until the column is filled up to M. Proceed with the next numbers in the next inner square in the same way, and when we arrive at the inner square of 3, write the magic square of 3

			M						
	1	2	3	4					
		17	18	19					6
			29	30				21	7
			44	37	42	32	22	8	
M	9	23	33	39	41	43				M
	16	28	36	40	45	38				
	15	27	35			34	31			
	14	26				24	25	20		
	13					10	11	12	5	
			M						

Fig. 1.

			M						
77	1	2	3	4	72	71	70	69	
76	62	17	18	19	58	57	56	6	
75	61	51	29	30	48	47	21	7	
74	60	50	44	37	42	32	22	8	
M 9	23	33	39	41	43	49	59	73	M
16	28	36	40	45	38	46	54	66	
15	27	35	53	52	34	31	55	67	
14	26	65	64	63	24	25	20	68	
13	81	80	79	78	10	11	12	5	
			M						

Fig. 2.

in full as shown in Fig. 1. The square is now completed by subtracting each number from n^2+1 and placing the difference in the appropriate square as shown in Fig. 2.

Proof. The sum of the numbers in the first row taken from the sum of the numbers in the last row totals 41, and since there are

four blanks in the bottom row we see that the sum of the bottom row is $4(n^2+1)+41=369$, and the sum in the top row is

$$5(n^2+1)-41=369.$$

Hence the square is magic in top and bottom rows. A similar result occurs when we deal with the first and last columns, and so on.

Note that the central number is $\frac{1}{2}(n^2+1)=\frac{1}{2}(9^2+1)=41$. Hence the sum of the inner square is $3\times 41=123$, of the next $5\times 41=205$, of the next $7\times 41=287$, and of the large outer square $9\times 41=369$.

II. Even-Bordered Squares.

The rules for even-bordered squares are much more difficult, and for that reason we must deal more fully with them. There are two kinds of borders to be considered: (i) a border when the square or sub-square is a multiple of 4; (ii) a border when the square or sub-square is not a multiple of 4, such as 6, 10, 14, etc. But there are a few general points that apply to every square as guides for their construction.

We place the letter N in line with the last column but one of the innermost square, and the letter E in line with the last row of the

N

	1			4	5			8	9			12
		23			27		31		35		32	13
			41			44	45			48	26	14
	15	28		55			59		60	49		
					65	78	79	68	58	50	30	16
	17	34	51	62	72	75	74	69				
					76	71	70	73	63	52	36	18
E	19	38	53	64	77	66	67	80				
	20		54		56	57		61			39	
		40		42	43			46	47			21
	22		24	25		29		33		37		
		2	3			6	7			10	11	

Fig. 3.

innermost square. These two letters enable us to simplify the writing out of the rules. For simplicity we use two diagrams: in Fig. 3 the outermost border is doubly-even; * in Fig. 4 the outermost border is singly-even. †

Rules. When the border is a multiple of 4, write the first number in the top left-hand corner and then the numbers 2, 3; 4, 5; 6, 7; 8, 9; 10, 11 as shown in Fig. 3. Then 12, 13, 14; 15; 16; 17;

* Doubly-even when the square is a multiple of 4.

† Singly-even when the square is not a multiple of 4.

18 ; 19, 20 ; the two numbers being placed so that the first coincides with E.

Every doubly-even border is made in the same way, as note the third border in Fig. 3. Fig. 4 shows an outer border singly-even; note how it runs until we reach N where 9 is placed, and how when we reach N the next number is placed in the diagonal at the top right-hand corner. Another guide is that most of the numbers in the top and bottom row differ by 4. The next border beginning with 19 is doubly-even, and is constructed as already shown.

N

1			5		9		13		10
	19			22	23			26	4
6		33			37		38	27	
			43	56	57	46	36	28	8
12	29	40	50	53	52	47			
			54	49	48	51	41	30	14
E 16	31		55	44	45	58			
	32		34	35		39			17
18		20	21			24	25		
	2	3		7		11		15	

FIG. 4.

Each square is completed by subtracting each number from n^2+1 and placing the difference in the appropriate empty square that is either the extreme diagonal point or the extreme diagonal point opposite the number which we have subtracted from n^2+1.

In Fig. 3, $n^2+1=12^2+1=145$;

in Fig. 4, $n^2+1=10^2+1=101$.

The proof is easy, for we note that the sum of the numbers in the first and last rows of Fig. 3 is 39 in each case, while the sum of the first and last columns is in each case 94. J. T.

GLEANINGS FAR AND NEAR.

1281. I should like to know what is the origin of the use of the square root of -1 as a sign of the Fourth Dimension. Obviously to the non-mathematical mind the symbol is nonsense, for there is no number which, multiplied by itself, comes to -1. Is there any mathematical warrant for it? We are likely to meet it a good deal in the future, even in schoolbooks.—Letter to the Editor of *The Observer*, October 30, 1938, p. 11. [Per Mr. E. S. Pondiczery.]

1282. Throughout the history of the science of probability, professors have asserted that, in games of chance, the very improbable is impossible. . . . Modern mathematicians . . . say now that a thousand million to one is the same as impossibility.—Norwood Young, *Fortuna, or Chance and Design*, 1928, pp. 70-71. [Per Mr. E. S. Pondiczery.]

FORMULAE FOR NUMERICAL INTEGRATION.

By W. G. Bickley.

Numerical integration, including the numerical integration of differential equations, is becoming increasingly necessary in technical applications. For such purposes, when a very high order of accuracy is not required, formulae which use successive values of the function to be integrated are more convenient than those using differences. Individual authors have given formulae of this type from time to time but a systematic list is not known to the present writer, and would seem to be of service.

The present list gives formulae of four types: (I) formulae for evaluating definite integrals; (II) formulae for forward integration; (III) formulae for integrating over a subrange; and (IV) formulae involving values of both the function and its integral. The leading term in the error series is given. Finally, some comments are made upon the formulae and their uses.

Notation.

The function to be integrated is denoted by q, and its integral by y, so that, x being the independent variable,

$$q = dy/dx.$$

It is assumed that values of q and y are tabulated (or are being tabulated) at equidistant intervals of x; the tabular interval is denoted by h. If x_0 is a pivotal value of x, we denote $q(x_0 + rh)$ by q_r, and similarly for y. The general form of the formulae given in the first three groups is

$$y_m - y_0 = h \sum_0^n A_r^{(n)} q_r.$$

The error term contains a derivative of q, and the order of this is indicated by superscript Roman numerals; the argument of this derivative is some value of x between 0 and mh.

Group I.

Formulae for evaluating definite integrals.

One-strip.

$$(I.1) \qquad y_1 - y_0 = \frac{h}{2}(q_0 + q_1) \qquad [-q^{ii}h^3/12$$

(Trapezoidal rule.)

Two-strip.

$$(I.2) \qquad y_2 - y_0 = \frac{h}{3}(q_0 + 4q_1 + q_2) \qquad [-q^{iv}h^5/90$$

(Simpson's first, or One Third, rule.)

Three-strip.

$$(I.3) \qquad y_3 - y_0 = \frac{3h}{8}(q_0 + 3q_1 + 3q_2 + q_3) \qquad [-3q^{iv}h^5/80$$

(Simpson's second, or Three Eighths, rule.)

Four-strip.

$$(I.4) \qquad y_4 - y_0 = \frac{2h}{45}\{7(q_0 + q_4) + 32(q_1 + q_3) + 12q_2\}$$

$$[-8q^{vi}h^7/945$$

Five-strip.

$$(I.5) \qquad y_5 - y_0 = \frac{5h}{288}\{19(q_0 + q_5) + 75(q_1 + q_4) + 50(q_2 + q_3)\}$$

$$[-275q^{vi}h^7/12096$$

Six-strip.

$$(I.6) \qquad y_6 - y_0 = \frac{h}{140}\{41(q_0 + q_6) + 216(q_1 + q_5) + 27(q_2 + q_4) + 272q_3\}$$

$$[-9q^{viii}h^9/1400$$

Eight-strip.

$$(I.8) \qquad y_8 - y_0 = \frac{4h}{14175}\{989(q_0 + q_8) + 5888(q_1 + q_7) - 928(q_2 + q_6) + 10496(q_3 + q_5) - 4540q_4\}$$

$$[-2368q^{x}h^{11}/467775$$

Ten-strip.

$$(I.10) \qquad y_{10} - y_0 = \frac{5h}{299376}\{16067(q_0 + q_{10}) + 106300(q_1 + q_9) - 48525(q_2 + q_8) + 272400(q_3 + q_7) - 260550(q_4 + q_6) + 427368q_5\}$$

$$[-134635q^{xii}h^{13}/326918592$$

The above are formulae giving the maximum accuracy possible for the number of ordinates used. Other formulae are :

Four-strip.

$$(I.4a) \qquad y_4 - y_0 = \frac{h}{3}\{q_0 + q_4 + 4(q_1 + q_3) + 2q_2\} \qquad [-q^{iv}h^5/45$$

(Double Simpson.)

Six-strip.

$$(I.6a) \qquad y_6 - y_0 = \frac{3h}{10}\{q_0 + q_6 + 5(q_1 + q_5) + q_2 + q_4 + 6q_3\}$$

$$[-q^{vi}h^7/140$$

(Weddle's rule.)

(I.6b) $y_6 - y_0 = h\{0{\cdot}28(q_0 + q_6) + 1{\cdot}62(q_1 + q_5) + 2{\cdot}2q_3\}$

$[+9q^{vi}h^7/700$

(G. F. Hardy's rule.)

Ten-strip.

(I.10a) $y_{10} - y_0 = \frac{5h}{126}\{8(q_0 + q_{10}) + 35(q_1 + q_3 + q_7 + q_9) + 15(q_2 + q_4 + q_6 + q_8) + 36q_5\}$

$[-25q^{vi}h^7/756$

(Shovelton's rule.)

Group II.

Formulae for forward integration.

(II.0) $y_1 - y_0 = hq_0$ $[+q^{i}h^2/2$

One-strip.

(II.1) $y_2 - y_0 = 2hq_1$ $[+q^{ii}h^3/3$

(Modified Euler, or Mid-ordinate, rule.)

Two-strip.

(II.2) $y_3 - y_0 = \frac{3h}{4}(q_0 + 3q_2)$ $[+3q^{iii}h^4/8$

(Heun's rule.)

Three-strip.

(II.3) $y_4 - y_0 = \frac{4h}{3}(2q_1 - q_2 + 2q_3)$ $[+14q^{iv}h^5/45$

(Milne's rule.)

Four-Strip.

(II.4) $y_5 - y_0 = \frac{5h}{144}(19q_0 - 10q_1 + 120q_2 - 70q_3 + 85q_4)$

$[+95q^{v}h^6/288$

Five-strip.

(II.5) $y_6 - y_0 = \frac{3h}{10}(11q_1 - 14q_2 + 26q_3 - 14q_4 + 11q_5)$

$[+41q^{vi}h^7/140$

We may also add :

(II.2a) $y_4 - y_0 = \frac{4h}{3}(2q_0 - 4q_1 + 5q_2)$ $[+8q^{iii}h^4/3$

Group III.

Partial-range formulae.

Two-strip.

(III.2) $$y_1 - y_0 = \frac{h}{12}(5q_0 + 8q_1 - q_2) \qquad [+q^{iii}h^4/24$$

Three-strip.

(III.3.1) $$y_1 - y_0 = \frac{h}{24}(9q_0 + 19q_1 - 5q_2 + q_3) \qquad [-19q^{iv}h^5/720$$

(III.3.2) $$y_2 - y_0 = \frac{h}{3}(q_0 + 4q_1 + q_2) \qquad [-q^{iv}h^5/90$$

Four-strip.

(III.4.1) $$y_1 - y_0 = \frac{h}{720}(251q_0 + 646q_1 - 264q_2 + 106q_3 - 19q_4)$$

$$[+3q^{v}h^6/160$$

(III.4.2) $$y_2 - y_0 = \frac{h}{90}(29q_0 + 124q_1 + 24q_2 + 4q_3 - q_4)$$

$$[+q^{v}h^6/90$$

(III.4.3) $$y_3 - y_0 = \frac{3h}{80}(9q_0 + 34q_1 + 24q_2 + 14q_3 - q_4)$$

$$[+3q^{v}h^6/160$$

Coefficients for five- and six-strip formulae are included in the table given later.

Group IV.

Check formulae of high accuracy containing series of values of both y and q.

Two-strip.

(IV.2) $$y_2 - y_0 = h(q_0 + 4q_1 + q_2) \qquad [-q^{iv}h^5/90$$

Three-strip.

(IV.3) $$11(y_3 - y_0) + 27(y_2 - y_1) = 3h(q_3 + 9q_2 + 9q_1 + q_0)$$

$$[-3q^{vi}h^7/140$$

Four-strip.

(IV.4) $$25(y_4 - y_0) + 160(y_3 - y_1)$$
$$= 6h\{q_4 + q_0 + 16(q_3 + q_1) + 36q_2\}$$

$$[-q^{viii}h^9/105$$

Six-strip.

(IV.6) $$49(y_6-y_0)+924(y_5-y_1)+2625(y_4-y_2)$$
$$=10h\{q_6+q_0+36(q_5+q_1)+225(q_4+q_2)+400q_3\}$$
$$[-5q^{xii}h^{13}/6006$$

Similar formulae, of lower accuracy, are :

Two-strip.

(IV.2a) $y_2-2y_1+y_0=h(q_2-q_0)/2$ $[-q^{iii}h^4/12$

Three-strip.

(IV.3a) $3(y_3-y_2-y_1+y_0)=h(q_3+3q_2-3q_1-q_0)$
$[-q^{v}h^6/30$

Four-strip.

(IV.4*a*) $$11(y_4+y_0)+16(y_3+y_1)-54y_2$$
$$=3h\{(q_4-q_0)+8(q_3-q_1)\}$$
$$[-3q^{vii}h^8/140$$

Six-strip.

(IV.6a) $$137(y_6+y_0)+1488(y_5+y_1)+375(y_4+y_2)-4000y_3$$
$$=30h\{(q_6-q_0)+24(q_5-q_1)+75(q_4-q_2)\}$$
$$[-5q^{xi}h^{12}/462$$

A simple formula of this type is :

(IV.3b) $y_3-y_0+9(y_2-y_1)=6h(q_2+q_1)$ $[+q^{iv}h^5/10$

Remarks on the Formulae.

Group I.

The most surprising feature of this group is the fact that the formula for $(2n+1)$ strips is *less* accurate than that for $2n$ strips. This is the explanation of the relatively high accuracy which can, as is well known, be obtained from Simpson's (first, or one-third) rule. Simpson's other (three-eighths) rule has a percentage error (for approximately constant q^{iv}) twice as great. The formulae for seven and nine strips have, in consequence, been omitted.

The presence of negative coefficients (as in the formulae for eight or more strips) is a disadvantage, since this increases the possible rounding-off error.

As the number of strips increases, the coefficients soon become inconvenient. This fact has led to the construction of formulae such as those of G. F. Hardy, Weddle, and Shovelton, in which some accuracy has been sacrificed to obtain simpler coefficients. It may be noted, however, that the errors in the formulae of Hardy and Weddle (both six-strip formulae) are of opposite sign, so that they form a check upon each other, and results are certainly reliable

as far as they agree. Noting also that the errors are (if powers of h higher than h^7 are neglected) in the ratio $9 : -5$, it is seen that a combination in the ratio $5 : 9$ will be more accurate. Indeed, such a weighted mean is equivalent to the most accurate six-strip rule (I.6).

GROUP II.

This set of formulae is of particular use in the numerical integration of differential equations, since they allow y to be calculated ahead of the values of q already tabulated. The error terms given make it possible to form some estimate of the value of h necessary to secure a desired accuracy. It cannot, however, be too strongly emphasised that these "error" terms cannot give a reliable *a priori* estimate of the resultant error arising in a continuous step-by-step integration—even if an estimate of the appropriate derivative of q can be found—on account of cumulative error. Thus some independent check at each stage is necessary. The natural check formula is that of group I for the same m (but see below). In view of the remarks above upon group I, it is advisable that the check formula relate to an even number of strips. The increasing complexity of the coefficients, along with the appearance of negative ones, makes it seldom worth while to use more than four strips, so that the method advocated by Milne, using (II.3) for forward integration, is seen to be probably the best compromise between labour and accuracy.

In connection with Milne's forward integration formula, there is an advantage in employing first, as a check, the formula

$$y_4 - y_0 = 2h(q_0 + 4q_2 + q_4)/3 \qquad [-16q^{iv}h^5/45, \tag{I.4b}$$

since the error term is approximately equal to that of the Milne formula ($14q^{iv}h^5/45$) and of opposite sign. If the results disagree only slightly, a simple average may be accurate enough, but it will be found that a combination in the ratio $8 : 7$ is equivalent to the most accurate four-strip rule (I.4).

The Heun formula (II.2), although it does not lend itself to the use of an "even" check formula, has nevertheless one great merit. It permits the interval to be halved with facility at any stage. From q_{n-2} and q_n we find y_{n+1}, while from q_{n-1} and q_n we find $y_{n+1/2}$, and so obtain three values at interval $\frac{1}{2}h$.

GROUP III.

In commencing the systematic step-by-step integration of a differential equation, it is evident that the formulae of the preceding groups cannot be applied until a few of the earlier values have been obtained. The method usually advocated is the use of series but this is not always convenient, and in any case needs a subsidiary calculation to determine the necessary coefficients. It is also evident that the accuracy of the whole solution cannot be expected to be greater than that of these early values, upon which everything else

is based. The formulae of group III enable the early values to be obtained by a process of successive approximation, which usually converges rapidly. (If the convergence is not rapid, it is an indication that the interval should be reduced!) Starting with a reasonable approximation to q_1 and q_2 (q_0 being, of course, known) we use (III.2) and (I.2) to obtain approximations to y_1 and y_2. These lead to better approximations to q_1 and q_2, which in their turn lead by the same process to still better ones. We may now start integrating ahead, and return, say by the use of the set (III.4) to improve the accuracy of y_1, y_2, y_3 and y_4.

GROUP IV.

This is a set of formulae of high accuracy, which can be used to supplement other checks, and it is believed that most of them are new. Owing to the small coefficient of the leading y, they are not of much use for integration, because when used to determine this y rounding-off errors will have considerable effect.

DERIVATION.

A word or two upon the derivation of the formulae may not be out of place. The individual coefficients in groups I-III can be obtained independently as integrals of the appropriate Lagrange interpolation polynomial over the appropriate range. For the calculation of the whole set, however, the use of the Maclaurin series is preferable, since this leads to a set of simultaneous equations for the coefficients which can be solved in a convenient and systematic manner. Also, it is not difficult to show that for constant m, if

$$y_m - y_0 = \int_0^{mh} q\,dx = h\sum_0^n A_r^{(n)} q_r,$$

then

(A) $$A_r^{(n-1)} = A_r^{(n)} - (-)^{n-r}\,{}^nC_r A_n^{(n)},$$

and

(B) the leading term of the error is $A_{n+1}^{(n+1)} q^{(n)} h^{n+1}$, unless this vanishes, when we must replace $(n+1)$ by $(n+2)$. Finally, we give the formulae of groups I-III, and some additional formulae of these types, in tabular form, classified in a different manner. (A) and (B) are checks on the accuracy of this table, while (B) enables the error term to be obtained in those cases which have not been previously given.

I wish to acknowledge my indebtedness to my colleagues, Messrs. F. J. Turton and G. B. Ehrenborg, who, by drawing my attention to some of these formulae, led me to undertake the systematic investigation whose results are now given.

TABLE OF COEFFICIENTS

IN THE FORMULA

$$y_m - y_0 = hN \sum_0^n B_r q_r / D.$$

n	m	N	B_0	B_1	B_2	B_3	B_4	B_5	B_6	D
1	1	1	1	1						2
	2	2	0	1						1
2	1	1	5	8	−1					12
	2	1	1	4	1					3
	3	3	1	0	3					4
	4	4	2	−4	5					3
3	1	1	9	19	−5	1				24
	2	1	1	4	1	0				3
	3	3	1	3	3	1				8
	4	4	0	2	−1	2				3
4	1	1	251	646	−264	106	−19			720
	2	1	29	124	24	4	−1			90
	3	3	9	34	24	14	−1			80
	4	2	7	32	12	32	7			45
	5	5	19	−10	120	−70	85			144
5	1	1	475	1427	−798	482	−173	27		1440
	2	1	28	129	14	14	−6	1		90
	3	3	17	73	38	38	−7	1		160
	4	2	7	32	12	32	7	0		45
	5	5	19	75	50	50	75	19		288
	6	3	0	11	−14	26	−14	11		10
6	1	1	19087	65112	−46461	37504	−20211	6312	−863	60480
	2	1	1139	5640	33	1328	−807	264	−37	3780
	3	1	685	3240	1161	2176	−729	216	−29	2240
	4	2	143	696	192	752	87	24	−4	945
	5	5	743	3480	1275	3200	2325	1128	−55	12096
	6	1	41	216	27	272	27	216	41	140

1283. She (Countess Benckendorff) was impatient of A. E. Housman's Cherry Tree poem :

> And take from seventy springs a score,
> It only leaves me fifty more.

"I don't want to do mental arithmetic when I'm reading poetry" (this though she had a remarkable gift for mathematics). But Bertrand Russell had told me that great mathematicians were usually weak in their sums.—Sir Edward Marsh, *Memoirs, Sunday Times*, January 8, 1939. [From the Rev. E. M. Radford ; Mr. P. J. Harris.]

MR. GIBBINS' TRIANGLE.

By F. H. V. Gulasekharam.

In the *Gazette* No. 249, Vol. XXII, Mr. Gibbins has deduced some interesting formulae connected with the triangle ABC, of which the side BC is given by $L \equiv lx + my + n = 0$ and AB, AC by the equation

$$S(x, y) \equiv S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

The equations of the circles in § 3 of Mr. Gibbins' note are very elegant indeed. But, in principle, it does not seem desirable to have recourse to pure geometry in dealing with problems in algebraic geometry. The algebraic analysis is not difficult. It is not necessary to resolve S into factors. Let us *assume* that $S \equiv vw$, where $v \equiv l_2x + m_2y + n_2$ and $w \equiv l_3x + m_3y + n_3$. Then we note that

$$ax^2 + 2hxy + by^2 \equiv (l_2x + m_2y)(l_3x + m_3y);$$

$$2X \equiv 2(ax + hy + g) \equiv \frac{\partial S}{\partial x} \equiv l_3v + l_2w;$$

$$2Y \equiv 2(hx + by + f) \equiv \frac{\partial S}{\partial y} \equiv m_3v + m_2w.$$

With these identities all Mr. Gibbins' main results can be easily deduced.

1. *Mensuration of the triangle.*

The formulae given by Mr. Gibbins for $\tan A$ and $2R \sin B \sin C$ and those deduced from them are not accurate.

If $P(x, y)$ be any point *in* BC, then $\tan A = \pm 2(h^2 - ab)^{\frac{1}{2}}/(a + b)$, according as $(l_2l_3 + m_2m_3)vw \lesseqgtr 0$—that is, according as $(a + b)S \lesseqgtr 0$.

When does P lie in BC?

If PBC makes an angle θ with the positive direction of the x-axis, then the lengths $\overrightarrow{PB}$, $\overrightarrow{PC}$ are the values of r given by the equation

$$S(x + r\cos\theta,\ y + r\sin\theta) = 0.$$

Hence $\overrightarrow{PB} \cdot \overrightarrow{PC} = S/(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$, a result which is true for any conic $S = 0$.

Since $$\tan\theta = -l/m,$$

we have $$\overrightarrow{PB} \cdot \overrightarrow{PC} = S(l^2 + m^2)/(am^2 - 2hlm + bl^2).$$

Hence the condition that P should lie *in* BC is

$$S(am^2 - 2hlm + bl^2) < 0.$$

Hence

(i) if $am^2 - 2hlm + bl^2 > 0$, then $S < 0$, and

$$\tan A = 2\mu(h^2 - ab)^{\frac{1}{2}}/(a + b);$$

(ii) if $am^2 - 2hlm + bl^2 < 0$, then $S > 0$, and

$$\tan A = -2\mu(h^2 - ab)^{\frac{1}{2}}/(a + b),$$

where $\mu = \pm 1$, according as $a+b \gtrless 0$.

Again we should note that

$$2R \sin B \sin C = \pm L_0/\surd(l^2+m^2),$$

according as $L_0 \gtrless 0$, where $A \equiv (x_0, y_0)$ and $L_0 \equiv lx_0 + my_0 + n$.

2. *The circle of which BC is a diameter.*

If $L=0$ cuts *any conic* $S=0$ in $B(x_1, y_1)$ and $C(x_2, y_2)$, the equation of the circle of which BC is a diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0,$$

where x_1, x_2 are the roots of the equation

$$m^2 S[x, -(lx+n)/m] = 0.$$

Hence $m^2 S[x, -(lx+n)/m] \equiv (am^2 - 2hlm + bl^2)(x-x_1)(x-x_2)$.

Similarly

$$l^2 S[-(ly+n)/l, y] \equiv (am^2 - 2hlm + bl^2)(y-y_1)(y-y_2).$$

Hence the equation of the circle is

$$m^2 S[x, -(lx+n)/m] + l^2 S[-(ly+n)/l, y] = 0.$$

If we write

$$-(lx+n) = my - L, \quad \text{and} \quad -(my+n) = lx - L,$$

the equation of the circle becomes

$$m^2 S\left[x, y - \frac{L}{m}\right] + l^2 S\left[x - \frac{L}{l}, y\right] = 0,$$

i.e.
$$m^2\left[S - 2\frac{L}{m}Y + b\frac{L^2}{m^2}\right] + l^2\left[S - 2\frac{L}{l}X + a\frac{L^2}{l^2}\right] = 0,$$

i.e.
$$(l^2+m^2)S - 2L(lX+mY) + (a+b)L^2 = 0.$$

If the preceding method seems *artificial*, the following method may be recommended.

Let $P_1(x_1, y_1)$, $P(x, y)$ be any two points. Then any other point on P_1P is $[(x_1+\lambda x)/(1+\lambda), (y_1+\lambda y)/(1+\lambda)]$.

If this point lies on the *conic* $S=0$,

$$\lambda^2 S + 2\lambda S_1 + S_{11} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where $S_{11} = S(x_1, y_1)$,

and
$$\begin{aligned} S_1 &= \Sigma a x_1 x + \Sigma h(xy_1 + x_1 y) \\ &= x_1 X + y_1 Y + gx + fy + c \\ &= xX_1 + yY_1 + gx_1 + fy_1 + c. \end{aligned}$$

If the point lies on $L=0$, then

$$L_1 + \lambda L = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

where $L_1 = lx_1 + my_1 + n$.

Hence if the straight line P_1P passes through either B or C, we get, on eliminating λ from equations (i) and (ii),

$$L_1^2 S - 2L_1 L S_1 + S_{11} L^2 = 0,$$

which being the relation satisfied by the coordinates of any point (x, y) on either of the lines P_1B, P_1C, is the equation of those lines.

Hence the necessary and sufficient condition that BC should subtend a right angle at P_1 is

$$(a+b)L_1^2 - 2L_1(lX_1 + mY_1) + (l^2+m^2)S_{11} = 0.$$

From this we see at once that the equation of the circle of which BC is a diameter is

$$(a+b)L^2 - 2L(lX+mY) + (l^2+m^2)S = 0.$$

NOTE. The circle cuts the conic $S=0$ again at the points

$$S = 0,$$

$$2(lX+mY) - (a+b)L = 0.$$

3. *The altitudes through B and C.*

If $D_1 = l_2l_3 + m_2m_3$, $D_2 = l_3l + m_3m$, and $D_3 = l_2l + m_2m$, it is easy to see that the altitudes through B and C are given by the equation

$$(D_1L - D_2v)(D_1L - D_3w) = 0,$$

i.e.
$$D_1^2L^2 - D_1L(D_2v + D_3w) + D_2D_3vw = 0.$$

We note that
$$D_1 = (a+b),$$

$$D_2D_3 = al^2 + 2hlm + bm^2,$$

and
$$D_2v + D_3w = l(l_2w + l_3v) + m(m_2w + m_3v)$$
$$= 2(lX + mY).$$

Hence the equation of the altitudes becomes

$$(a+b)^2L^2 - 2(a+b)L(lX+mY) + (al^2 + 2hlm + bm^2)S = 0.$$

4. *The medians through B and C.*

If $C_1 = l_2m_3 - l_3m_2$, $C_2 = l_3m - m_3l$, and $C_3 = lm_2 - ml_1$, the medians through B and C are given by the equation

$$(C_1L - C_2v)(C_1L - C_3w) = 0,$$

i.e.
$$C_1^2L^2 - C_1L(C_2v + C_3w) + C_2C_3vw = 0.$$

We note that
$$C_1^2 = 4(h^2 - ab),$$

$$C_2C_3 = (l_3m - m_3l)(lm_2 - l_2m)$$
$$= -(am^2 - 2hlm + bl^2).$$

[In the identity

$$(l_2x + m_2y)(l_3x + m_3y) = ax^2 + 2hxy + by^2,$$

put $x = m$, $y = -l$.]

Again, $C_2v + C_3w = 0$ is the straight line through A parallel to BC. Hence $C_2v + C_3w$ must be of the form $k(L - L_0)$. It is readily seen that $k = -C_1$. Thus the equation of the two medians becomes

$$4(h^2 - ab)L(2L - L_0) = (am^2 - 2hlm + bl^2)S.$$

5. *The bisectors of the angles at B and C.*

The single equation is

$$\left(\frac{L^2}{l^2+m^2}-\frac{v^2}{l_2{}^2+m_2{}^2}\right)\left(\frac{L^2}{l^2+m^2}-\frac{w^2}{l_3{}^2+m_3{}^2}\right)=0,$$

i.e. $(l_2{}^2+m_2{}^2)(l_3{}^2+m_3{}^2)L^4$

$$-(l^2+m^2)L^2[v^2(l_3{}^2+m_3{}^2)+w^2(l_2{}^2+m_2{}^2)]+(l^2+m^2)^2v^2w^2=0.$$

We have $(l_2{}^2+m_2{}^2)(l_3{}^2+m_3{}^2)=(a-b)^2+4h^2.$

Again, since $$l_3v+l_2w=\frac{\partial S}{\partial x}=2X,$$

and $$m_3v+m_2w=\frac{\partial S}{\partial y}=2Y,$$

we have $v^2(l_3{}^2+m_3{}^2)+w^2(l_2{}^2+m_2{}^2)$

$$=4(X^2+Y^2)-2(l_2l_3+m_2m_3)vw$$

$$=4(X^2+Y^2)-2(a+b)S.$$

Thus the equation of the four bisectors becomes

$$[(a-b)^2+4h^2]L^4$$

$$-2(l^2+m^2)L^2[2(X^2+Y^2)-(a+b)S]+(l^2+m^2)S^2=0.$$

F. H. V. GULASEKHARAM.

1284. The intellectual pleasure that matters most to him [Mr. De Valera] is Mathematics. One day going to Rome he asked his secretary what he thought of the quaternary theorem. Nothing, the secretary replied, who knew only elementary mathematics. It was a boiling hot day but Dev spent twelve solid hours teaching his secretary the quaternary theorem. The secretary said that Dev's twelve hour lecture was the most brilliant intellectual performance he had ever known.—John Gunther, *Inside Europe.* [Per Mr. R. J. A. Barnard.]

1285. Mathematicians were especially impressed by the unique scoop of 1892. That summer the Problem Paper of the Maths. Tripos was set on a Friday morning. The "Granta" came out on Saturday. It would be a smart piece of journalism to publish, within 24 hours, the full solution of the Problem Paper in a Special Supplement. As soon as the questions were available to the public—at 9.30 a.m. on the Friday—the paper was rushed off to Clare, where two high wranglers were in readiness. All day they worked, practically locked in a room till all the questions were solved—except one. It beat them, and the scheme looked like being ruined—till a man happened to drop in who had done the problem in the examination!

After that it was plain sailing. The printers were kept at work on the Supplement till five a.m.; at 10.30 it was on sale in the streets. Moreover, a copy found its way to the breakfast table of each of the examiners, with a note to the effect that complaints that the Mathematical Tripos was becoming year by year more difficult appeared groundless. The editor of the "Granta" had merely called in his office boy, who had knocked off the Problem Paper and gone on with his work!—Article on the *Granta* Jubilee, *Daily Telegraph*, March 7, 1939.

TRIANGLES ÉQUILATÉRAUX DÉRIVÉS D'UN TRIANGLE QUELCONQUE.

PAR GINO LORIA,
Professeur à l'Université de Gênes.

ON a récemment signalé des simples constructions qui mènent d'un triangle plan tout à fait arbitraire à un triangle regulier. On peut établir que ce triangle dérivé jouit de cette remarquable qualité en s'inspirant à une profonde maxime d'Abel (" il faut énoncer un problème de manière de le pouvoir toujours résoudre "); on doit alors se proposer en général de déterminer de quelle espèce est le triangle résultant de l'application d'une des constructions citées. Dans plusieurs cas il arrive que l'expression de la longueur d'un quelconque de ses côtés est une fonction symétrique des éléments du triangle primitif; alors la régularité du triangle d'arrivée est hors de doute. Il est aussi à remarquer qu'à l'expression cherchée on arrive très souvent par un procédé (qui rappelle la " triangulation géodésique ") qui mérite d'être signalée à tous ceux qui auront à l'avenir à s'occuper de questions analogues à celles étudiées dans le mémoire qui suit.*

Les constructions rappelées au début sont de deux espèces différentes; par conséquent nous nous en occuperons séparément.

I.

1. Sur les côtés d'un triangle ABC (Fig. 1) dont les côtés seront désignés par les lettres a, b, c on construit *extérieurement* les triangles équilatéraux BCA', CAB', ABC', et on en considère les centres L', M', N'; de quelle espèce est le triangle $L'M'N'$?

Pour répondre à cette question fixons notre attention sur le triangle auxiliaire $AM'N'$; on a évidemment :

$$AM' = b/\sqrt{3}, \quad AN' = c/\sqrt{3}, \quad \angle M'AN' = A + \tfrac{1}{3}\pi.$$

Par conséquent

$$M'N'^2 = \tfrac{1}{3}\{b^2 + c^2 - 2bc\cos(A + \tfrac{1}{3}\pi)\}$$
$$= \tfrac{1}{3}\{b^2 + c^2 - bc\cos A + \sqrt{3}bc\sin A\}.$$

Or
$$bc\cos A = \tfrac{1}{2}(b^2 + c^2 - a^2),$$
$$\tfrac{1}{2}bc\sin A = S, \text{ aire du triangle } ABC;$$

donc
$$M'N'^2 = \tfrac{1}{6}(a^2 + b^2 + c^2) + S/2\sqrt{3}.$$

Comme cette expréssion est symétrique par rapport à a, b, c on voit que $M'N' = N'L' = L'M'$ et on est autorisé à conclure ce théorème :

Les centres des trois triangles équilatéraux construits extérieurement

* Qu'il me soit permis de remarquer que je m'intéresse depuis longtemps aux types de raisonnements qu'on peut employer dans la recherche mathématique; j'ai résumé mes conclusions dans le volume *Metodi Matematici-Essenza, Tecnica, Applicazioni* (Milan, Hoepli, 1935).

*sur les côtés d'un triangle quelconque sont les sommets d'un nouveau triangle équilatéral.**

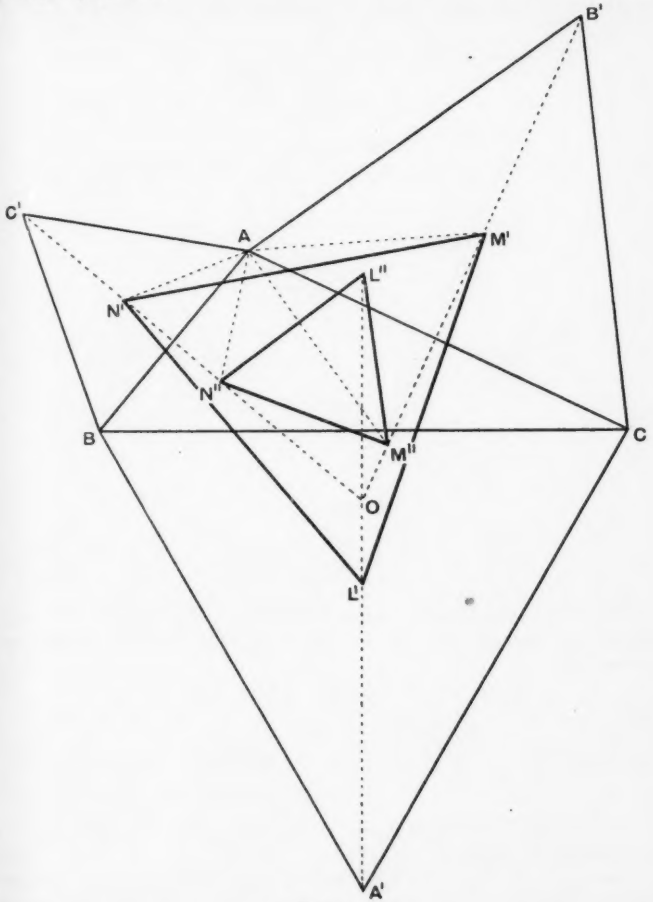


FIG. 1.

2. Il se présente naturellement la question de savoir si on arrive à la même conclusion on construisant *intérieurement* trois triangles équilatéraux BCA'', CAB'', ABC'' (même figure) sur les côtés du triangle ABC, supposé toujours d'une espèce quelconque. Appelons L'', M'', N'' les centres de ces triangles et remarquons qu'ils sont

* Cette proposition a été enoncée dans le cahier de Juin 1938 du journal *Sphinx* (p. 106).

les symétriques des points L', M', N' par rapport aux côtés du triangle donné, les droites $L'L''$, $M'M''$, $N'N''$ passant par le circumcentre O du triangle donné, de manière que les triangles $L'M'N'$

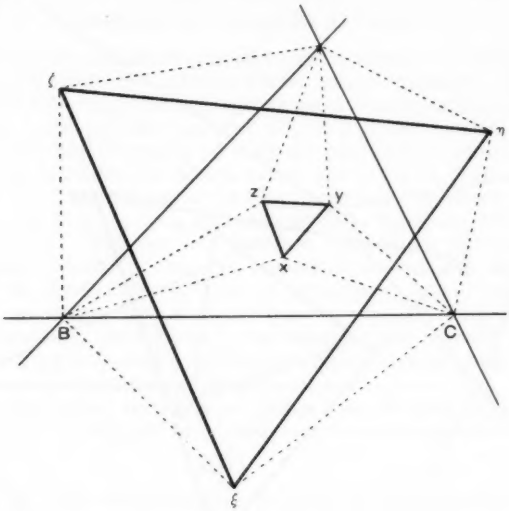


FIG. 2.

et $L''M''N''$ sont homologiques par rapport à ce centre. En considérant le triangle $AM''N''$ on remarque tout de suite les relations suivantes :

$$AM'' = AM' = b/\sqrt{3}, \quad AN'' = AN' = c/\sqrt{3},$$
$$\angle M''AN'' = |A - \tfrac{1}{3}\pi|$$

et ensuite
$$M''N''^2 = \tfrac{1}{3}\{b^2 + c^2 - 2bc\cos(A - \tfrac{1}{3}\pi)$$
$$= \tfrac{1}{6}(a^2 + b^2 + c^2) - S/2\sqrt{3}.$$

Comme on trouverait la même expréssion pour $N''L''$ et $L''M''$ on est autorisé à conclure :

Les centre des triangles équilatéraux construits intérieurement *sur les côtes d'un triangle quelconque sont les sommets d'un nouveau triangle équilatéral.**

* Il est naturel de se poser la question si des théorèmes analogues ont lieu en construisant sur les côtés du triangle donné des polygones réguliers d'un nombre de côtés plus grand que trois ; le cas de carrés montre que la chose n'est pas possible ; car, par un raisonnement tout à fait analogue à ceux que nous venons d'exposer, on arrive à la conclusion que les côtés du triangles resultant ne sont pas égaux entre eux, car leurs carrés sont exprimés comme il suit :

$$\tfrac{1}{2}(b^2 + c^2) + 2S, \quad \tfrac{1}{2}(c^2 + a^2) + 2S, \quad \tfrac{1}{2}(a^2 + b^2) + 2S.$$

II.

3. La seconde méthode de dérivation de triangles équilatéraux est bien plus féconde que celle que nous venons d'exposer ; elle a sa base dans la considération, introduite dans la science par un mathématicien américain bien connu, F. Morley, des droites trisectrices des angles d'un triangle quelconque.*

Pour procéder du simple au composé je m'occupe avant tout du cas où l'on considère les angles intérieurs du triangle donné ABC (Fig. 2). Les premières trisectrices des angles B et C se coupent au point x ; une signification analogue ont les lettres y et z ; or de quel espèce est le triangle xyz? Pour répondre à cette question on peut employer un raisonnement du tout analogue à celui que nous avons employée précédemment. Remarquons, en effet, qu'on a :

$$\angle BxC = \pi - \tfrac{1}{3}(B+C) = \tfrac{2}{3}\pi + A ;$$

en conséquence le triangle BCx nous donne :

$$a/\sin(\tfrac{2}{3}\pi + A) = Bx/\sin\tfrac{1}{3}C$$

donc (R rayon du circumcercle)

$$Bx = \frac{a\sin\frac{1}{3}C}{\sin(\frac{2}{3}\pi + A)} = \frac{2R\sin A\sin\frac{1}{3}C}{\sin(\frac{2}{3}\pi + A)} .$$

Mais, ayant en général,

$$\sin\tfrac{1}{3}x\sin\tfrac{1}{3}(\pi+x)\sin\tfrac{1}{3}(2\pi+x) = \tfrac{1}{4}\sin x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

on peut écrire

$$Bx = 8R\sin\tfrac{1}{3}A\sin\tfrac{1}{3}(\pi+A)\sin\tfrac{1}{3}C.$$

Par l'échange de A avec C on tire de cette formule cette autre

$$Bz = 8R\sin\tfrac{1}{3}C\sin\tfrac{1}{3}(\pi+C)\sin\tfrac{1}{3}A.$$

* Un document important pour l'histoire de cette question est représente par la lettre suivante qui est ici publiée pour la première fois d'après l'original qui se trouve chez moi :

6 HAYMARKET,
LONDON, 22 *Aug.* 1934.

MY DEAR PROFESSOR LORIA,

The exact source is F. Morley in *Mathematical Association of Japan for Secondary Mathematics*, Vol. 6, Dec. 1924.

I mentioned the theorem also in " Extension of Clifford's Chain Theorem ", *American Journal of Mathematics*, Vol. 51, July 1929.

I mentioned the theorem to friends in Cambridge, England, around 1904, but as it was a part of a theory I did not make a paper of it. Thus it became public, and the Japanese reference is *not* I think the first reference in print. I am very sorry for the long delay in answering, but I am you see in England.—Yours with high regards,

FRANK MORLEY.

Le sujet a désormais une littérature assez étendue ; j'ai eu l'occasion d'examiner seulement les publications suivantes : G. B. Zucca, *Periodico di matematiche*, IV Ser., T. i, 1921, p. 291 ; (comp. mes *Metodi Matematici* déjà cités, pp. 58-61). J. Neuberg, Sur les trisectrices des angles d'un triangle (*Mathésis*, T. xxxvii, 1923, pp. 356-367). H. v. Kaven, Ein Satz über die Winkeldreiteilenden im Dreieck (*Zeitschr. f. math. und naturw. Unterricht*, T. lxix, 1938, pp. 155-157). J. E. Hoffmann, Ueber die Figur der Winkeldreiteilenden im Dreieck (*Ib.* pp. 158-162). W. J. Dobbs, Morley's triangle (*The Math. Gazette*, February 1938, pp. 50-57).

J'adopte dans la suite toutes les notations employées dans ce beau mémoire, que je suppose parfaitement connu par mes lecteurs.

Or on a

$$(xz)^2 = (Bx)^2 + (Bz)^2 - 2(Bx)(Bz)\cos\tfrac{1}{3}B\,; \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

donc

$$\frac{(xz)^2}{64\sin^2\tfrac{1}{3}A\sin^2\tfrac{1}{3}C} = \sin^2\tfrac{1}{3}(\pi+A) + \sin^2\tfrac{1}{3}(\pi+C) - 2\sin\tfrac{1}{3}(\pi+A)\sin\tfrac{1}{3}(\pi+C)\cos\tfrac{1}{3}B. \quad \ldots\text{(iii)}$$

Pour transformer le second membre de cette relation on peut avoir recours à l'identité suivante :

$$m^2 + n^2 - 2mn\cos\phi = (m-n)^2\cos^2\tfrac{1}{2}\phi + (m+n)^2\sin^2\tfrac{1}{2}\phi.$$

En supposant en particulier

$$m = \sin\tfrac{1}{3}(\pi+A), \quad n = \sin\tfrac{1}{3}(\pi+C), \quad \phi = \tfrac{1}{3}B,$$

on trouve

$$m+n = 2\sin\tfrac{1}{6}B\sin\tfrac{1}{6}(A-C), \quad m-n = 2\cos\tfrac{1}{3}B\cos\tfrac{1}{6}(A-C).$$

Cela prouve que le second membre de la formule (iii) vaut simplement $\sin^2\frac{1}{3}B$; en substituant dans la relation (iii) et en extrayant les racines carrées des deux membres on conclut

$$xz = 8R\sin\tfrac{1}{3}A\sin\tfrac{1}{3}B\sin\tfrac{1}{3}C.$$

Comme cette expréssion est symétrique par rapport aux angles A, B, C, on voit qu'on serait arrivé au même résultat en calculant les autres côtés xy et yz ; donc le triangle xyz est équilatéral, conclusion dans laquelle consiste précisément le théorème de Morley.

4. On peut se poser la question si il y a des constructions analogues à celle qui a mené au triangle xyz donnant comme résultats de nouveaux triangles équilatéraux. Pour repondre, observons qu'en général quand on connait les côtés (droites indéfinies) d'un angle, cet angle n'est pas déterminé sans ambiguité car, si on appelle ϕ la mésure du plus petit des angles résultants, les autres seront mésurés par $\phi+\pi$ et $\phi+2\pi$. Il s'ensuit * que comme on a (n. 3) étudié le triangle xyz déterminé par les droites formant intérieurement avec les côtés du triangle donné ABC les angles $\frac{1}{3}A$, $\frac{1}{3}B$, $\frac{1}{3}C$ on pourra considérer : (a) (Fig. 2) le triangle $\xi\eta\zeta$ déterminé également par les droites formant avec les mêmes côtés extérieurement les angles $\frac{1}{3}(\pi-A)$, $\frac{1}{3}(\pi-B)$, $\frac{1}{3}(\pi-C)$, (b) (Fig. 3) ou bien le triangle XYZ déterminé par les droites analogues formant les angles $\frac{1}{3}(\pi-2A)$, $\frac{1}{3}(\pi-2B)$, $\frac{1}{3}(\pi-2C)$. Or on peut prouver que ces triangles sont eux aussi équilatéraux et en même temps déterminer les longueurs de leurs côtés, en employant le même raisonnement et presque les mêmes calculs qui nous permirent d'établir la régularité du triangle xyz.

Sans exposer tout au long ce raisonnement et ces calculs nous nous bornons a transcrire les formules relatives en suivant le même ordre que nous suivimes dans la recherche précédente :

* Pour plus de détails à ce sujet, voyez le mémoire de M. Dobbs.

(*a*) $\angle B\xi C = \pi - \frac{1}{3}\{2\pi - (B+C)\} = \frac{1}{3}(2\pi - A)$;

$\angle \xi B\zeta = \frac{1}{3}(2\pi + B)$.

$$\frac{B\xi}{\sin\frac{1}{3}(\pi - C)} = \frac{a}{\sin\frac{1}{3}(2\pi - A)} = \frac{2R\sin A}{\sin\frac{1}{3}(2\pi - A)} = \frac{2R\sin A}{\sin\frac{1}{3}(\pi + A)},$$

$B\xi = 8R\sin\frac{1}{3}A\sin\frac{1}{3}(2\pi + A)\sin\frac{1}{3}(2\pi + C)$;

$B\zeta = 8R\sin\frac{1}{3}C\sin\frac{1}{3}(2\pi + C)\sin\frac{1}{3}(2\pi + A)$.

$\xi\zeta^2/\{64R^2\sin^2\frac{1}{3}(2\pi + A)\sin^2\frac{1}{3}(2\pi + C)\}$

$= \sin^2\frac{1}{3}A + \sin^2\frac{1}{3}C - 2\sin\frac{1}{3}A\sin\frac{1}{3}C\cos\frac{1}{3}(2\pi + B)$

$= \sin^2\frac{1}{3}(2\pi + B)$;

$\xi\zeta = 8R\sin\frac{1}{3}(2\pi + A)\sin\frac{1}{3}(2\pi + B)\sin\frac{1}{3}(2\pi + C)$.

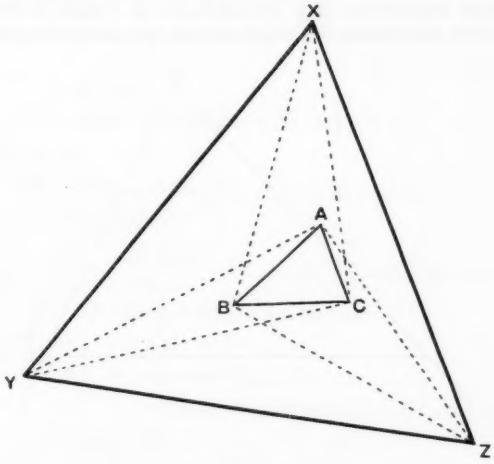


FIG. 3.

(*b*) $\angle BXC = \pi - \{\frac{2}{3}\pi + \frac{1}{3}(B+C)\} = \frac{1}{3}A$;

$\angle XBC = B + \frac{1}{3}\pi - \frac{2}{3}B = \frac{1}{3}(\pi + B)$.

$$\frac{BX}{\sin\frac{1}{3}(\pi + C)} = \frac{a}{\sin\frac{1}{3}A} = \frac{2R\sin A}{\sin\frac{1}{3}A}$$

$= 8R\sin\frac{1}{3}(\pi + A)\sin\frac{1}{3}(2\pi + A)$,

$BX = 8R\sin\frac{1}{3}(\pi + A)\sin\frac{1}{3}(2\pi + A)\sin\frac{1}{3}(\pi + C)$;

$BZ = 8R\sin\frac{1}{3}(\pi + C)\sin\frac{1}{3}(2\pi + C)\sin\frac{1}{3}(\pi + A)$.

$XZ^2/\{64R^2\sin^2\frac{1}{3}(\pi + A)\sin^2\frac{1}{3}(\pi + C)\}$

$= \sin^2\frac{1}{3}(2\pi + A) + \sin^2\frac{1}{3}(2\pi + C)$

$- 2\sin\frac{1}{3}(2\pi + A)\sin\frac{1}{3}(2\pi + C)\cos\frac{1}{3}(2\pi - B)$

$$= \sin^2 \tfrac{1}{3}(\pi + B),$$

$$XZ = 8R \sin \tfrac{1}{3}(\pi + A) \sin \tfrac{1}{3}(\pi + B) \sin \tfrac{1}{3}(\pi + C).$$

5. Il y a d'autres triangles auxquels mène la considération des trisectrices des angles du triangle ABC (voyez toujours le mémoire de M. Dobbs) qui sont équilatéraux quoique leurs côtés aient des expréssions qui ne sont pas symétriques par rapport aux éléments de ce triangle ; leur régularité peut s'établir par des calculs analogues à ceux que nous venons d'exposer, mais un peu plus longs, car il ne suffit pas de déterminer la longueur d'un des côtés ; pour n'abuser pas de l'hospitalité qui m'accorde *The Mathematical Gazette* je me bornerai à m'occuper d'un des triangles dont il s'agit, en exposant les calculs relatifs sous la forme schématique dont je me suis servi dans le n. 4.

Les deux trisectrices (Fig. 4) intérieurs de l'angle A rencontrent les premières bisectrices des angles formés par les prolongements des

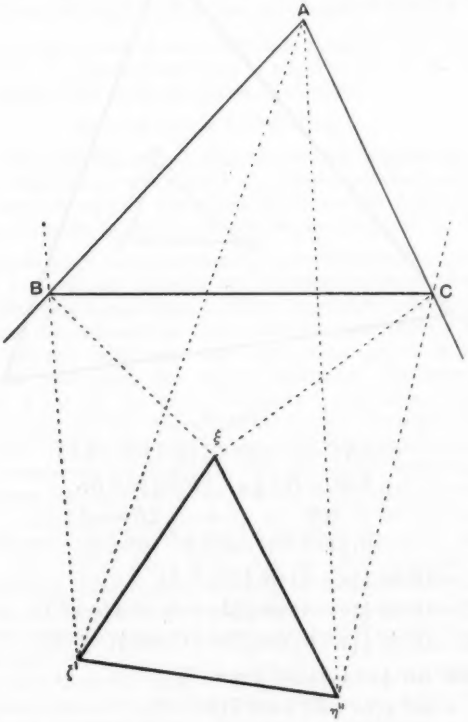


FIG. 4.

côtés AB et AC avec le côté BC aux points ζ' et η'' ; ces points avec le point ξ déjà considéré, forment un triangle que nous nous proposons d'étudier.

Calcul du côté $\zeta'\eta''$.

$$\angle AB\zeta' = B + \tfrac{2}{3}(\pi - B) = \tfrac{1}{3}(2\pi + B),$$

$$\angle A\zeta' B = \pi - \tfrac{1}{3}\{2\pi + (A+B)\} = \tfrac{1}{3}C.$$

$$\frac{A\zeta'}{\sin\frac{1}{3}(2\pi+B)} = \frac{AB}{\sin\frac{1}{3}C} = \frac{2R\sin C}{\sin\frac{1}{3}C} = 8R\sin\tfrac{1}{3}(\pi+C)\sin\tfrac{1}{3}(2\pi+C).$$

$$A\zeta' = 8R\sin\tfrac{1}{3}(2\pi+B)\sin\tfrac{1}{3}(\pi+C)\sin\tfrac{1}{3}(2\pi+C).$$

$$A\eta'' = 8R\sin\tfrac{1}{3}(2\pi+C)\sin\tfrac{1}{3}(\pi+B)\sin\tfrac{1}{3}(2\pi+B).$$

$$(\zeta'\eta'')^2/\{64R^2\sin\tfrac{1}{3}(2\pi+B)\sin^2\tfrac{1}{3}(2\pi+C)\}$$

$$= \sin^2\tfrac{1}{3}(\pi+B) + \sin^2\tfrac{1}{3}(\pi+C) - 2\sin\tfrac{1}{3}(\pi+B)\sin\tfrac{1}{3}(\pi+C)\cos\tfrac{1}{3}A$$

$$= \sin^2\tfrac{1}{3}A.$$

$$\zeta'\eta'' = 8R\sin\tfrac{1}{3}A\sin\tfrac{1}{3}(2\pi+B)\sin\tfrac{1}{3}(2\pi+C).$$

Calcul du côté $\xi\zeta'$.

$$B\xi = 8R\sin\tfrac{1}{3}A\sin\tfrac{1}{3}(2\pi+A)\sin\tfrac{1}{3}(2\pi+C),$$

(d'aprés un calcul précédent).

$$\frac{B\zeta'}{\sin\frac{1}{3}A} = \frac{AB}{\sin\frac{1}{3}C} = \frac{2R\sin C}{\sin\frac{1}{3}C} = 8R\sin\tfrac{1}{3}(\pi+C)\sin\tfrac{1}{3}(2\pi+C).$$

$$B\zeta' = 8R\sin\tfrac{1}{3}A\sin\tfrac{1}{3}(\pi+C)\sin\tfrac{1}{3}(2\pi+C).$$

$$(\xi\zeta')^2/\{64R^2\sin^2\tfrac{1}{3}A\sin\tfrac{1}{3}(2\pi+C)\}$$

$$= \sin^2\tfrac{1}{3}(2\pi+A) + \sin^2\tfrac{1}{3}(\pi+C) - 2\sin\tfrac{1}{3}(2\pi+A)\sin\tfrac{1}{3}(\pi+C)\cos\tfrac{1}{3}(\pi-B)$$

$$= \sin^2\tfrac{1}{3}(2\pi+B).$$

$$\xi\zeta' = 8R\sin\tfrac{1}{3}A\sin\tfrac{1}{3}(2\pi+B)\sin\tfrac{1}{3}(2\pi+C).$$

Comme cette expression est symétrique par rapport aux angles B et C, elle appartient aussi au côté $\xi\eta''$ du triangle considéré, et comme c'est la même que nous avons trouvé pour le côté $\zeta'\eta''$, on est en droit de conclure la régularité du triangle $\xi\zeta'\eta''$.

6. Un procédé parfaitement analogue peut s'appliquer à tous les 27 triangles auxquels mène la considération des trisectrices d'un triangle. Dans le cours de cette application on a besoin de calculer au préalable tous les angles de la figure complète dont il s'agit et en conséquence d'en avertir les propriétés, dont quelques-unes sont dignes de remarque ; par exemple on peut observer que les angles BXC, $B\xi'C$, $B\xi''C$ ont tous la valeur $\frac{1}{3}A$; il s'ensuit que les cinq points B, C, X, ξ', ξ'' se trouvent sur la même circonférence ; la même chose arrive par rapport aux groupes C, A, Y, η', η'' et A, B, Z, ζ', ζ''. Encore, comme les angles $\xi'CB$ et $\xi''BC$ ont tous les

deux comme valeur $\frac{2}{3}\pi$, les points B, C, ξ', ξ'' sont aussi concycliques, etc. Il faut encore avertir qu'on a lieu d'appliquer l'identité

$$\sin^2 x + \sin^2 (x + \tfrac{2}{3}\pi) + \sin x \,.\, \sin (x + \tfrac{2}{3}\pi) = \tfrac{3}{4},$$

qui a lieu quel que soit l'angle x.

Les résultats auxquels on arrive par les calculs dont il s'agit sont résumés dans le tableau suivant :

*Désignation des triangles.**	*Longueurs des côtés.*
xyz	$l_1 = 8R \sin \frac{1}{3}A \,.\, \sin \frac{1}{3}B \,.\, \sin \frac{1}{3}C.$
$\xi\eta\zeta$	$l_2 = 8R \sin (\frac{1}{3}A + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}B + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}C + \frac{1}{6}\pi).$
XYZ	$l_3 = 8R \sin (\frac{1}{3}A + \frac{1}{3}\pi) \,.\, \sin (\frac{1}{3}B + \frac{1}{3}\pi) \,.\, \sin (\frac{1}{3}C + \frac{1}{3}\pi).$
$\xi\eta''\zeta'$	$l_4 = 8R \sin \frac{1}{3}A \,.\, \sin (\frac{1}{3}B + \frac{1}{3}\pi) \,.\, \sin (\frac{1}{3}C + \frac{1}{3}\pi).$
$\xi'\eta\xi'$	$l_4' = 8R \sin (\frac{1}{3}A + \frac{1}{3}\pi) \,.\, \sin \frac{1}{3}B \,.\, \sin (\frac{1}{3}C + \frac{1}{3}\pi).$
$\xi''\eta'\zeta$	$l_4'' = 8R \sin (\frac{1}{3}A + \frac{1}{3}\pi) \,.\, \sin (\frac{1}{3}B + \frac{1}{3}\pi) \,.\, \sin \frac{1}{3}C.$
$xy''z'$	$l_5 = 8R \sin (\frac{1}{3}A + \frac{1}{6}\pi) \,.\, \sin \frac{1}{3}B \,.\, \sin \frac{1}{3}C.$
$x'yz''$	$l_5' = 8R \sin \frac{1}{3}A \,.\, \sin (\frac{1}{3}B + \frac{1}{6}\pi) \,.\, \sin \frac{1}{3}C.$
$x''y'z$	$l_5'' = 8R \sin \frac{1}{3}A \,.\, \sin \frac{1}{3}B \,.\, \sin (\frac{1}{3}C + \frac{1}{6}\pi).$
$XY''Z'$	$l_6 = 8R \sin (\frac{1}{3}A + \frac{1}{3}\pi) \,.\, \sin (\frac{1}{3}B + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}C + \frac{1}{6}\pi).$
$X'YZ''$	$l_6' = 8R \sin (\frac{1}{3}A + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}B + \frac{1}{3}\pi) \,.\, \sin (\frac{1}{3}C + \frac{1}{6}\pi).$
$X''Y'Z$	$l_6'' = 8R \sin (\frac{1}{3}A + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}B + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}C + \frac{1}{3}\pi).$
$X'y'\zeta''$	$l_7 = 8R \sin \frac{1}{3}A \,.\, \sin (\frac{1}{3}B + \frac{1}{3}\pi) \,.\, \sin (\frac{1}{3}C + \frac{1}{6}\pi).$
$Y'z'\xi''$	$l_7' = 8R \sin (\frac{1}{3}A + \frac{1}{6}\pi) \,.\, \sin \frac{1}{3}B \,.\, \sin (\frac{1}{3}C + \frac{1}{3}\pi).$
$Z'x'\eta''$	$l_7'' = 8R \sin (\frac{1}{3}A + \frac{1}{3}\pi) \,.\, \sin (\frac{1}{3}B + \frac{1}{6}\pi) \,.\, \sin \frac{1}{3}C.$
$X''\eta''z''$	$l_8 = 8R \sin \frac{1}{3}A \,.\, \sin (\frac{1}{3}B + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}C + \frac{1}{3}\pi).$
$Y''\zeta''x''$	$l_8' = 8R \sin (\frac{1}{3}A + \frac{1}{3}\pi) \,.\, \sin \frac{1}{3}B \,.\, \sin (\frac{1}{3}C + \frac{1}{6}\pi).$
$Z''\xi''y''$	$l_8'' = 8R \sin (\frac{1}{3}A + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}B + \frac{1}{3}\pi) \,.\, \sin \frac{1}{3}C.$
$X\xi''\xi''$	$l_9 = 4\sqrt{3}R \,.\, \sin (\frac{1}{3}A + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}A + \frac{1}{3}\pi).$
$Y\eta'\eta''$	$l_9' = 4\sqrt{3}R \,.\, \sin (\frac{1}{3}B + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}B + \frac{1}{3}\pi).$
$Z\zeta'\zeta''$	$l_9'' = 4\sqrt{3}R \,.\, \sin (\frac{1}{3}C + \frac{1}{6}\pi) \,.\, \sin (\frac{1}{3}C + \frac{1}{3}\pi).$
$\xi x'x''$	$l_{10} = 4\sqrt{3}R \sin \frac{1}{3}A \,.\, \sin (\frac{1}{3}A + \frac{1}{3}\pi).$
$\eta y'y''$	$l_{10}' = 4\sqrt{3}R \sin \frac{1}{3}B \,.\, \sin (\frac{1}{3}B + \frac{1}{3}\pi).$
$\zeta z'z''$	$l_{10}'' = 4\sqrt{3}R \sin \frac{1}{3}C \,.\, \sin (\frac{1}{3}C + \frac{1}{3}\pi).$
$xX'X''$	$l_{11} = 4\sqrt{3}R \,.\, \sin \frac{1}{3}A \,.\, \sin (\frac{1}{3}A + \frac{1}{6}\pi).$
$yY'Y''$	$l_{11}' = 4\sqrt{3}R \,.\, \sin \frac{1}{3}B \,.\, \sin (\frac{1}{3}B + \frac{1}{6}\pi).$
$zZ'Z''$	$l_{11}'' = 4\sqrt{3}R \,.\, \sin \frac{1}{3}C \,.\, \sin (\frac{1}{3}C + \frac{1}{6}\pi).$

Entre les longueurs que nous venons de calculer il y a une foule de relations qu'il est aisé de trouver, car il s'agit de 27 quantités qui sont des fonctions rationnelles et entières de neuf autres ; nous nous bornons de signaler la suivante : $l_1 l_2 l_3 = abc$.

* Je suppose que le lecteur ait sous les yeux la Fig. 8 du mémoire de M. Dobbs (*The Math. Gazette*, No. 248, p. 56).

TWO KINDS OF ELECTRICITY.

By M. Sperling.

Very often it is proved experimentally that two electrified bodies repel or attract each other, and from this fact is deduced the conclusion that there exist two kinds of electricity of such properties that bodies of the same kind of electricity repel and bodies of different kinds attract each other. This proceeding is not quite correct.

Example. If we consider two integral numbers, then their difference is divisible by 3 or not. But it is obviously wrong to deduce from this fact that we can divide the set of all integral numbers into two such classes, that the difference of two numbers belonging to one class is divisible through 3 and the difference of two numbers belonging to different classes is not. It is known that there exist three classes possessing these properties.

The purpose of this note is to give the minimal number of facts, which must be stated experimentally, in order to conclude that there exist only two kinds of electricity.

I

Let us consider a set of electrified bodies. To each of them we impute electricities called relatively A, B, C, ... We shall say that a bullet from elder pith (hanging on a silken thread) touched by a body with electricity A, has electricity A. If we touch a bullet having electricity A with a bullet having electricity B, there occurs one of two possibilities: either the bullets will repel or they will attract each other, which we shall denote respectively by $A \longleftrightarrow B$ or $A \rightarrow\!\leftarrow B$. That with other results of experience we can write down as follows:

In the set S whose elements are A, B, C, ... are defined two relations ($\longleftrightarrow$, $\rightarrow\!\leftarrow$), with the following properties:

1. Every pair (A, B) of elements belonging to S satisfies one and only one of the relations:

$$A \longleftrightarrow B, \quad A \rightarrow\!\leftarrow B.$$

2. There exist in the set S two elements A_0, B_0, for which

$$A_0 \rightarrow\!\leftarrow B_0.$$

3. If $A \longleftrightarrow B$, $A \longleftrightarrow C$, then $B \longleftrightarrow C$.

4. If $A \rightarrow\!\leftarrow B$, $A \rightarrow\!\leftarrow C$, then $B \longleftrightarrow C$.

From these four assumptions some consequences follow.

5. $A \longleftrightarrow A$.

Proof. If this relation were not true, then from 1 we would obtain $A \rightarrow\!\leftarrow A$. Then placing in 4 the element A instead of B and C, we have $A \longleftrightarrow A$, in contradiction to $A \rightarrow\!\leftarrow A$.

6. If $A \longleftrightarrow B$, then $B \longleftrightarrow A$.

Proof. Acting upon assumption $A \longleftrightarrow B$ and using $A \longleftrightarrow A$ (5), we get (from 3) $B \longleftrightarrow A$.

7. THEOREM. *The set S is a sum of two non-empty subsets $\{+\}$ and $\{-\}$ without common elements such that for every pair of elements A and B belonging to one of the subsets $A \longleftrightarrow B$, and for every pair of elements A and B belonging to different subsets $A \rightarrow \leftarrow B$.*

PROOF. It follows from 3, 5, 6 that S is a sum of subsets without common elements such that for two elements belonging to one subset $A \longleftrightarrow B$, and for two elements belonging to different subsets this relation does not hold.

Let $\{+\}$ denote the subset containing A_0. In S there exist elements not belonging to $\{+\}$. For instance, B_0 has this property; otherwise we should have $A_0 \longleftrightarrow B_0$, in contradiction to 1 and 2.

Let A be an element not belonging to $\{+\}$. $A_0 \longleftrightarrow A$ being not true, we have, by 1, $A_0 \rightarrow \leftarrow A$. From that relation, and from 2 and 4, we conclude $A \longleftrightarrow B_0$, *i.e.* A belongs to the same subset as B_0. This subset we denote by $\{-\}$. Then $S = \{+\} + \{-\}$.

Since for two elements of different subsets the relation $\longleftrightarrow$ does not hold, it follows from 1 that for these elements we must have $\rightarrow \leftarrow$. Our theorem is thus proved.

It is known that the electricity belonging to the same subset as the electricity of ebonite rubbed by fox's fur is called negative, and the electricity of the second subset is called positive.

II

We have seen that assumptions 1, 2, 3, 4 suffice to prove 7. It can be easily shown that also from 7 the conditions 1, 2, 3, 4 follow.

For instance, we shall show it for condition 4. Indeed, if $A \rightarrow \leftarrow B$ and $A \rightarrow \leftarrow C$, then by 7 B and C do not belong to the subset containing A; therefore they must belong to the second subset and from that follows $B \longleftrightarrow C$. (Q.E.D.)

Now we shall show that conditions 1, 2, 3, 4 are independent—that is, that none of them follows from the three remaining. Because conditions 1, 2, 3, 4 are equivalent to 7 we can enunciate the theorem as follows:

THEOREM. *No set of three from conditions* 1, 2, 3, 4 *suffices to prove* 7.

PROOF. Let S be the set of natural numbers. Let, further, $A \longleftrightarrow B$ denote that $A + B$ is an even number, and $A \rightarrow \leftarrow B$ that AB is an odd number.

The condition 1 is not true, because, for instance, $1 + 3$ is an even number and $1 \cdot 3$ is an odd number.

The condition 2 is fulfilled, for instance, by the numbers 1 and 3 ($1 \cdot 3$ is an odd number).

The condition 3 is true. If, namely, $A + B$ and $A + C$ are even numbers, then $B + C$ is also an even number.

The condition 4 we prove as follows. If AB and AC are odd numbers, *i.e.* A, B, C are odd, then $B + C$ is even.

From this construction it follows that 1 is not a consequence of the conditions 2, 3, 4.

Therefore 7 cannot be true in this set. It can also be verified directly:

For every pair of elements belonging to the subset of even or to the subset of odd numbers the relation $\longleftrightarrow$ is fulfilled, while for an even and an odd number neither $\longleftrightarrow$ nor $\rightarrow\leftarrow$ is true.

To prove the independence of 2 (or 7) from 1, 3, 4 it is sufficient to consider the set of natural numbers, where $A\longleftrightarrow B$ denotes $A+B\geqq 2$ and $A\rightarrow\leftarrow B$ denotes $A+B<2$ (the whole set of natural numbers is the unique subset defined by the relation $\longleftrightarrow$).

Now we take the set of all integral numbers and define $A\longleftrightarrow B$, $A\rightarrow\leftarrow B$ as $AB\geqq 0$, $AB<0$ respectively. We see that 3 is not true (*e.g.* $0\,.\,(-1)\geqq 0$, $0\,.\,2\geqq 0$, but $(-1\,.\,)2<0$), while 1, 2, 4 are fulfilled.

7 does not hold because every number ought to belong to the same subset as 0 ($0\,.\,A\geqslant 0$).

In order to prove the independence of 4 from the first three conditions we can consider the set of all natural numbers, in which $A\longleftrightarrow B$ denotes that $(A+1)\,(B+1)$ is odd (*i.e.* A and B are even), and $A\rightarrow\leftarrow B$ denotes that $(A+1)\,(B+1)$ is even. Then 4 is not true, for example, for numbers 1, 2, 3, while conditions 1, 2, 3 are fulfilled.

According to 7 we see that, for example, no two numbers from 1, 2, 3 fulfil the relation $\longleftrightarrow$.

Relations 1, 2, 3, 4 are not in a contradiction, if we assume that there is no contradiction in the set of natural numbers. Denoting, namely, by $A\longleftrightarrow B$ that $A+B$ is even, and by $A\rightarrow\leftarrow B$ that $A+B$ is odd, we see that the axioms 1, 2, 3, 4 have a realisation in the set of natural numbers.

In this case the subsets of the theorem 7 are the sets of even and odd numbers.

M. SPERLING (Lwów).

1286. As a writer, in the pulpit, on the platform, his [Dean Stanley's] brilliant gifts were streaked by a vein of helplessness in practical matters which was always appealing and sometimes amusing. I do not think that, to the day of his death, he understood the difference between eighteenpence and one and eightpence.—Lord Ernle, *Whippingham to Westminster*. [Per Miss M. M. Rogers.]

1287. But for anyone who knows what they are talking about, the very mathematicians are men of letters—men of algebraic letters, no doubt; but even x and sigma and psi can be aesthetically good or bad, litteræ humaniores or inhuman letters. I have heard mathematicians groaning over the demonstrations of Kelvin. Ponderous and clumsy, they bludgeon the mind into a reluctant assent. Whereas to be convinced by Clerk Maxwell's elegant equations is a pleasure; and reading Niels Abel on hyperelliptic functions is almost, it seems, like listening to Mozart's chamber music. For the mathematically illiterate, like myself, these things are, of course, mere scribblings, without significance and without form.—Aldous Huxley, *The Olive Tree and other Essays*, pp. 56-7, 1936, London. [Per Mr. F. Robbins.]

NOTES ON THE THEORY OF EQUATIONS.

By E. P. Lewis.

I. The Cubic

$$ax^3+3bx^2+3cx+d=0.$$

Multiply throughout by a^2 and write y for $ax+b$; the equation becomes

$$y^3+3Hy+G=0, \quad \text{.....(i)}$$

where $H\equiv ac-b^2$, $G\equiv a^2d-3abc+2b^3$.

Since in an equation with real coefficients complex roots occur in conjugate pairs, (i) must have at least one real root; so if α is this root, (i) may be written

$$(y-\alpha)(y^2+\alpha y+3H+\alpha^2)=0.$$

Accordingly the two remaining roots are also real if

$$\alpha^2+4H\leqslant 0.$$

But since α satisfies (i),

$$G^2=\alpha^2(\alpha^2+3H)^2$$

and so

$$\begin{aligned} G^2+4H^3&=\alpha^6+6H\alpha^4+9H^2\alpha^2+4H^3\\ &=(\alpha^2+4H)(\alpha^2+H)^2. \end{aligned}$$

Hence if (i) has three real roots, $G^2+4H^3\leqslant 0$; and clearly, when $G^2+4H^3=0$, two roots are numerically equal to $\surd(-H)$ and the third to $2\surd(-H)$.

It follows that when all three roots of (i) are real no one of them can exceed $2\surd(-H)$ in absolute value.

To find inferior limits when all three roots are real, let the roots be α, β, γ, so that

$$\Sigma\alpha=0, \quad \Sigma\beta\gamma=3H, \quad \alpha\beta\gamma=-G.$$

Then

$$\beta\gamma-3H=-\alpha(\beta+\gamma)=\alpha^2>0$$

and

$$\begin{aligned} 3(\beta\gamma+H)&=4\beta\gamma+\alpha(\beta+\gamma)\\ &=4\beta\gamma-(\beta+\gamma)^2\\ &=-(\beta-\gamma)^2, \end{aligned}$$

so that

$$\beta\gamma+H<0.$$

Now by Descartes' rule of signs, (i) has either one positive and two negative roots or one negative and two positive roots; hence if $\beta\gamma$ is positive, α and G have opposite signs but

$$-G/\alpha+H<0.$$

Hence

$$|\alpha|>|G|/(-H).$$

If $\beta\gamma$ is negative, α and G have the same sign and

$$-G/\alpha-3H>0$$

or

$$|\alpha|>|G|/(-3H).$$

Since α, β, γ are interchangeable, both upper and lower limits have been found for the roots when all are real.

It is easy to show that when (i) has only one real root α,

if $\quad H<0, \quad |G|/(-H)>|\alpha|>2\sqrt{(-H)}$;

if $\quad H>0, \quad |\alpha|<|G|/3H.$

Consider next the case when the three roots are real and two of them, β and γ say, are nearly equal. Since

$$3(\beta\gamma+H)=-(\beta-\gamma)^2,$$

β and γ are both approximately equal to $\sqrt{(-H)}$ with the appropriate sign and α is approximately equal to $2\sqrt{(-H)}$ with the opposite sign, so $G^2+4H^3\simeq 0$. Conversely, when G^2+4H^3 is negative and numerically small, two of the roots of (i) will be approximately equal.

Illustration. (Burnside and Panton, *Theory of Equations*, 5th edition, I, p. 208, No. 7.)

The equation $\quad x^3+11x^2-102x+181=0 \quad$ .....(A)

becomes $\quad y^3+3Hy+G=0,$

where $\quad H=-427/9, \quad G=17647/27, \quad G^2+4H^3=-49/27.$

For the largest root, α, negative in this case,

$$2\sqrt{(-H)}>|\alpha|>|G|/(-H),$$

giving $\quad 13{\cdot}77598\ldots>|\alpha|>13{\cdot}77595.$

Thus the numerically largest root of (A) is $-17{\cdot}4426$ very nearly, while the nearly equal roots are both in the neighbourhood of $3{\cdot}2$. These can be determined to the same degree of accuracy quite simply. Diminish the roots of (A) by $3{\cdot}2$; the resulting equation,

$$y^3+20{\cdot}6y^2-0{\cdot}88y+0{\cdot}008=0,$$

has two very small roots, so y^3 is negligible for this pair and the residual quadratic gives the values $0{\cdot}0295\ldots$, $0{\cdot}0131\ldots$. The three roots of (A) are thus $3{\cdot}2295\ldots$, $3{\cdot}2131\ldots$, $-17{\cdot}4426$.

II. The Biquadratic

$$ax^4+4bx^3+6cx^2+4dx+e=0.$$

Multiply throughout by a^3 and write y for $ax+b$; the equation becomes

$$y^4+6Hy^2+4Gy+a^2I-3H^2=0, \quad \text{.....(1)}$$

where $H\equiv ac-b^2$, $G\equiv a^2d-3abc+2b^3$, $I\equiv ae-4bd+3c^2$. As usual, let $J\equiv ace+2bcd-ad^2-b^2e-c^3$, then it is easy to verify that

$$G^2+4H^3-a^2HI\equiv -a^3J.$$

If the roots of (i) are α, β, γ, δ,

$$\Sigma\alpha=0 \quad \text{and} \quad 6H=\alpha\beta+\gamma\delta+(\alpha+\beta)(\gamma+\delta),$$

whence $$6H+(\alpha+\beta)^2=\alpha\beta+\gamma\delta.$$

Also $$-4G=\gamma\delta(\alpha+\beta)+\alpha\beta(\gamma+\delta)=(\gamma\delta-\alpha\beta)(\alpha+\beta)$$

and so $$4G/(\alpha+\beta)=\alpha\beta-\gamma\delta.$$

For brevity write p for $\alpha+\beta$, then

$$2\alpha\beta=p^2+6H+4G/p,$$

$$2\gamma\delta=p^2+6H-4G/p\ ;$$

therefore $$4(a^2I-3H^2)=4\alpha\beta\gamma\delta$$

$$=(p^2+6H)^2-16G^2/p^2,$$

so $$p^6+12Hp^4+4p^2(12H^2-a^2I)-16G^2=0. \quad \text{.....(ii)}$$

This may be written

$$(p^2+4H)^3-4a^2I(p^2+4H)=16G^2+64H^3-16a^2HI,$$

so that $$(p^2+4H)^3-4a^2I(p^2+4H)+16a^3J=0. \quad \text{.....(iii)}$$

The condition that this as a cubic in p^2+4H may have all its roots real is $I^3-27J^2\geqslant 0$, involving $I>0$ unless $I=J=0$. The roots in p^2 of (ii) will then be either one positive and two negative, in which case all four roots of (i) will be complex, or three positive, in which case all four roots of (i) will be real. The further conditions necessary for this are :

$$H<0, \quad 12H^2-a^2I>0.$$

If $I^3-27J^2<0$, two of the roots of (iii) will be complex and so two values of p^2 from (ii) will be complex, and this can only occur when equation (i) has two real and two complex roots.

If two roots of equation (i) are equal, say $\alpha=\beta$, equation (iii) has two equal roots also, and for the third $p^2=4\alpha^2$, in this case $I^3=27J^2$ and

$$4(\alpha^2+H)^3-a^2I(\alpha^2+H)+a^3J=0.$$

If $I\neq 0$, this last equation may be written :

$$\{\alpha^2+H+(3aJ)/I\}\{2(\alpha^2+H)-(3aJ/I)\}^2=0.$$

The squared factor refers to the repeated root of the cubic so

$$\alpha^2+H=-3aJ/I,$$

but from (i) α also satisfies the equation

$$\alpha^3+3H\alpha+G=0.$$

Hence $$\alpha\{2H-(3aJ/I)\}=-G,$$

whence $\alpha=GI/(3aJ-2HI)$ if $G\neq 0$.

Also, the squared factor gives

$$\{\tfrac{1}{2}(\alpha+\gamma)^2+2H\}=3aJ/I$$

or $$(\alpha+\gamma)^2=2(3aJ-2HI)/I.$$

Hence if two roots of the equation (i) are equal, the roots are

$$\alpha, \quad \alpha, \quad -\alpha\pm\sqrt{\{2(3aJ-2HI)/I\}},$$

where $$\alpha=GI/(3aJ-2HI).$$

It is clear that all four roots will be real if $3aJ-2HI>0$, and in this case α will have the same sign as G.

But, if $3aJ-2HI=0$, then $G=0$, and (i) has two pairs of equal roots, namely $\pm\sqrt{(-3H)}$, $\pm\sqrt{(-3H)}$.

If (i) has three equal roots, (iii) has also three equal roots; hence $I=J=0$ and the four roots of (i) are

either $\quad +\sqrt{(-H)}$ (three times), $-3\sqrt{(-H)}$

or $\quad -\sqrt{(-H)}$ (three times), $+3\sqrt{(-H)}$.

When the four roots of (i) are real and unequal, a value of p^2 can be determined from (iii) either accurately or approximately and the complete solution of (i) can then be obtained by making use of the fact that

$$y^4+6Hy^2+4Gy+a^2I-3H^2\equiv(y^2+3H+\tfrac{1}{2}p^2)^2-(py-2G/p)^2.$$

III. The General Equation.

If in the general equation

$$a_0x^n+na_1x^{n-1}+{}^nC_2a_2x^{n-2}+\ldots=0$$

y is written for a_0x+a_1, the equation after multiplication by $a_0{}^{n-1}$ becomes

$$y^n+{}^nC_2Hy^{n-2}+{}^nC_3Gy^{n-3}+\ldots=0, \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where $\quad H\equiv a_0a_2-a_1{}^2, \quad G\equiv a_0{}^2a_3-3a_0a_1a_2+2a_1{}^3.$

Assuming that (i) has roots α, β, γ, ..., the equation may be written

$$(y-\alpha)\{y^{n-1}+\alpha y^{n-2}+({}^nC_2H+\alpha^2)y^{n-3}+\ldots\}=0.$$

So $\quad \sum_{n=1}\beta=-\alpha, \quad \sum_{n-1}\beta\gamma={}^nC_2H+\alpha^2.$

Now
$$\begin{aligned}\sum_{n-1}(\beta-\gamma)^2&=(n-2)\sum_{n-1}\beta^2-2\sum_{n-1}\beta\gamma\\&=(n-2)\{(\sum_{n-1}\beta)^2-2\sum_{n-1}\beta\gamma\}-2\sum_{n-1}\beta\gamma\\&=(n-2)(\sum_{n-1}\beta)^2-(2n-2)\sum_{n-1}\beta\gamma\\&=(n-2)\alpha^2-(2n-2)\{{}^nC_2H+\alpha^2\}\\&=-n\{\alpha^2+(n-1)^2H\}.\end{aligned}$$

Hence for equation (i) to have n real roots H must be negative and $\alpha^2+(n-1)^2H$ must also be negative, where α is any one of the real roots. Hence no one of the n real roots of (i) can exceed $(n-1)\sqrt{(-H)}$ in absolute magnitude. Conversely, if any root of (i) exceeds $(n-1)\sqrt{(-H)}$ in absolute value the equation cannot have all its roots real. E. P. L.

1288. The General Electric Company has produced vacuums which were 99,999,999 per cent. perfect, but they admit that this is considerably less than a perfect vacuum.—*Parade*, July 1939, p. 63. [Per Prof. R. O. Street.]

MATHEMATICAL NOTES.

1398. *Triangles with sides and medians commensurable.*

Denoting the sides by a, b, c and the corresponding medians by l, m, n, we have

$$a^2+b^2=2\{(\tfrac{1}{2}c)^2+n^2\}.$$

Putting
$$a=h+k, \quad b=h-k,$$
$$a^2+b^2=2(h^2+k^2),$$

but this gives only a flat triangle so we must find two other numbers u and v such that

$$h^2+k^2=u^2+v^2 \quad \text{or} \quad h^2-u^2=v^2-k^2;$$

and putting

$$h+u=2px, \quad h-u=2qy,$$
$$v+k=2qx, \quad v-k=2py,$$

we have

$$h=px+qy, \quad k=qx-py,$$

and
$$\begin{aligned} a^2+b^2&=\{(p+q)x-(p-q)y\}^2+\{(p-q)x+(p+q)y\}^2 \\ &=2(p^2+q^2)(x^2+y^2) \\ &=2(px+qy)^2+2(qx-py)^2 \\ \text{or} \quad &=2(px-qy)^2+2(qx+py)^2 \end{aligned}\Big\}.$$

The first alternative gives the flat triangle, the second may give a solution. Thus

$$1^2+13^2=2(6^2+7^2)=2(2^2+9^2)=7^2+11^2,$$

where 2 and 9 for half-base and median fail to form a triangle with sides 1 and 13 but 6 and 7 yield two triangles with sides 7 and 11.

To find a second commensurable median: if

$$a=(p+q)x-(p-q)y,$$
$$b=(p-q)x+(p+q)y,$$
$$c=2(qx+py),$$

then
$$n=px-qy,$$
$$4l^2=\{(p-3q)x+(3p+q)y\}^2+32pqxy,$$
$$4m^2=\{(p+3q)x+(q-3p)y\}^2+32pqxy;$$

hence
$$4l^2-\{(p-3q)x+(3p+q)y\}^2=32pqxy,$$

so we may write

$$2l+\{(p-3q)x+(3p+q)y\}=32pqxy/2s,$$
$$2l-\{(p-3q)x+(3p+q)y\}=2s,$$

whence
$$(p-3q)x+(3p+q)y=8pqxy/s-s,$$

where s is any number. Thus, putting

$$p=2, \quad q=1, \quad s=8x,$$

we find $7x+5y=0$

or $x=5, \quad y=-7$

and $a=22, \quad b=-16, \quad c=-18, \quad l=13, \quad n=17.$

(The minus signs may be disregarded.)

In the same way, if m is also to be commensurable,

$$(p+3q)x+(q-3p)y=8pqxy/r-r,$$

and a common solution must be found for this and the last equation. By trial I find that

$$s=4qy, \quad r=8qx$$

gives $p=29, \quad q=1, \quad x=23, \quad y=8,$

so that we have

$$a=466, \quad b=884, \quad c=510,$$

$$l=683, \quad m=208, \quad n=659$$

and $s=2py, \quad r=px$

gives $a=982, \quad b=932, \quad c=1614,$

$$l=1223, \quad m=1252, \quad n=515.$$

Other trials with similar simple values for s and r give the same triangles and I should like to know of others with smaller numbers for the lengths of their sides. C. H. HARDINGHAM.

1399. *To draw a square of which a given parallelogram is the orthogonal projection.*

The problem considered in Notes 940, *Gazette*, XIV, p. 462; 1248, XXI, p. 283; 1356, XXIII, p. 89 consists in finding the isosceles right-angled triangle $B_1B_2B_3$ (right angle at B_1) having as projection a given triangle $A_1A_2A_3$.

The more general problem, to find a triangle similar to a given triangle $D_1D_2D_3$ and having as projection another given triangle $A_1A_2A_3$, has been solved before in a very elegant form by Neuberg as a generalisation of a configuration considered by Lionnet * for the special case when $D_1D_2D_3$ is equilateral (Neuberg, "Sur les projections et contre-projections d'un triangle fixe," *Mémoires de l'Académie royale des Sciences de Belgique*, 1890).

Neuberg's theorems are as follows:

If externally on the sides of triangle $A_1A_2A_3$ triangles $A_1'A_2A_3$, $A_1A_2'A_3$, $A_1A_2A_3'$, and internally on the same sides triangles $A_1''A_2A_3$, $A_1A_2''A_3$, $A_1A_2A_3''$, are constructed similar to $D_1D_2D_3$, so that homologous vertices correspond to the same indices, the bisectrices of the angles formed by A_1A_1' and A_1A_1'', A_2A_2' and A_2A_2'', A_3A_3' and A_3A_3'' are parallel to the intersections of the plane $A_1A_2A_3$ by the planes of the triangles $B_1B_2B_3$ or $C_1C_2C_3$

* *Nouvelles Annales de Mathématiques*, 1869, p. 528.

similar to $D_1D_2D_3$, having $A_1A_2A_3$ as projection or which is the projection of $A_1A_2A_3$.

If, further, E_1, E_2, E_3 and E_1', E_2', E_3' are the centres of the circles $A_1'A_2A_3$, $A_1A_2'A_3$, $A_1A_2A_3'$ and $A_1''A_2A_3$, $A_1A_2''A_3$, $A_1A_2A_3''$, the sides of $B_1B_2B_3$ and $C_1C_2C_3$, are

$$E_2E_3 \pm E_2'E_3', \quad E_3E_1 \pm E_3'E_1', \quad E_1E_2 \pm E_1'E_2'.$$

In the special case considered in the above-mentioned Notes, these constructions become very simple.

Let $H_1A_2H_1'A_3$, $H_2A_3H_2'A_1$, $H_3A_1H_3'A_2$ be the squares having the sides A_2A_3, A_3A_1, A_1A_2 as diagonals, the points H_1, H_2, H_3 being on the external side with respect to A_2A_3, A_3A_1, A_1A_2.

Then the bisectrices of $H_1A_1H_1'$ are the intersections of the plane $A_1A_2A_3$ by the planes of the right-angled triangles $B_1B_2B_3$ or $C_1C_2C_3$, having as projection $A_1A_2A_3$ or which is the projection of $A_1A_2A_3$.

When A_1 is obtuse, $B_1B_2B_3$ corresponds to the bisectrix of $H_1A_1H_1'$ external to $A_2A_1A_3$ and $C_1C_2C_3$ to the internal one; when A_1 is acute, $B_1B_2B_3$ corresponds to the bisectrix of $H_1A_1H_1'$ internal to $A_2A_1A_3$ and $C_1C_2C_3$ to the external one.

Further, the sides of the triangles $B_1B_2B_3$ and $C_1C_2C_3$ are

$$H_2H_3 \pm H_2'H_3', \quad H_3M_1 \pm H_3'M_1, \quad M_1H_2 \pm M_1H_2',$$

M_1 being the mid-point of A_2A_3.

R. GOORMAGHTIGH.

1400. *On Differentiation.*

This is a Note on the short article in the *Gazette* of October 1938, "On Differentiation from First Principles".* In agreeing generally on the essential point, I would add these comments:

(1) The only basic "algebraic" cases are the two uniquely simple cases of $y=x$ and y constant—yielding, respectively, $y'=1$ and $y'=0$.†

(2) No apology need be offered for treating the Circular Functions geometrically. These arise naturally as *continuous* trigonometric functions of a variable *angle*. For differentiation, the argument must be a *number* (not an *angle*) and $\sin x$, etc., therefore defined as the sine, etc., of the angle whose measure, in terms of *any* unit angle, is x. The limit-problem arising is then that of $(\sin x)/x$,‡ when $x \to 0$; and the standard geometrical inequalities lead directly to the value k for that limit—where k denotes the *circular measure of the unit angle* to which $\sin x$ refers. Whence, again, the derivatives of $\sin x$, $\cos x$, $\tan x$, etc., are $k.\cos x$, $-k.\sin x$, $k.\sec^2 x$, etc.

* XXII, 251, p. 374.

† See XIII, 190 (Oct. 1927), p. 412. Note the expression of these fundamental cases in terms of the basic numbers 0 and 1.

‡ Note the proof of these—bearing back very simply upon the definition of arc-length (see also Note 1208; XX, 240 (Oct. 1936), p. 276)—in XIII, 190, p. 414. At that time the k-limit and its implications were not clear in my mind.

These results are, of course, simplest when $k=1$ (this being, in fact, the key to the importance of circular measure in Trigonometry—as distinct from " mensuration " of the Circle) ; but it is not sufficiently realised how simple *and primary* (*not secondary*) the more general results—as stated—are.

(3) The arguments *against* defining $\log x$ by the simple definite integral are at least as strong as those *for* so defining it. However we proceed, $\log x$ must be correlated with $\log_a b$—including $\log_2 8=3$—and the " common logarithms " of our Mathematical Tables ; and there is no escaping all the implications of that correlation. A specially simple property of a function does not necessarily provide the best definition—however tempting. But this is far too large a question for discussion in a brief note.* For the moment it is sufficient to lodge a mild *caveat*. D. K. PICKEN.

1401. *On a certain type of partial differential equations.*

The standard form $F(p, q, z)=0$ is discussed in many textbooks by showing that it can be reduced to an *ordinary* differential equation by assuming a solution to exist in the form $z=f(x+ay)$. Instead of this, it will be easier to remember as a working rule that the equation can always be solved by associating with it the equation $p=aq$ and using $dz=pdx+qdy$. This is, in fact, what would be done by using Charpit's method. The geometrical meaning of the differential equation and this solution is that it is an equation that is satisfied by an ∞^2 of cylinders.

Let us consider *the most general partial differential equation of the first order in x, y, z which admits of an ∞^2 of cylinders for solution.* Any cylinder satisfies a differential equation of the form $ap+bq=c$, where a, b, c are constants. If $f(x, y, z, p, q)=0$ is consistent with $ap+bq=c$, we get from Charpit's condition

$$a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}+(ap+bq)\frac{\partial f}{\partial z}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If this is taken as a linear differential equation in the unknown f involving x, y, z, p, q, Charpit's subsidiary equations are

$$\frac{dx}{a}=\frac{dy}{b}=\frac{dz}{ap+bq}=\frac{dp}{0}=\frac{dq}{0}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Among the integrals of (2), we have $p=\text{const.}$, $q=\text{const.}$, $ay-bx=\text{const.}$, $az-x(ap+bq)=\text{const.}$, $bz-y(ap+bq)=\text{const.}$, and

$$px+qy-z=\text{const.}$$

Hence f may be taken in any one of several forms such as

$$f\equiv F_1(p, q, ay-bx, az-x\,.\,\overline{ap+bq})=0, \quad \ldots\ldots\ldots\ldots(3a)$$

* It is bound up with the subject of which " Commonsense of Number "—XXII, 250 (July 1938), p. 225—is a summary. There are more ways than one of ensuring that pupils will have " nothing to unlearn " at a later stage ; and it is not yet certain which of these ways is best.

$$f \equiv F_2(p, q, ay - bx, bz - y \,.\, \overline{ap + bq}) = 0, \quad \ldots\ldots\ldots\ldots (3b)$$

$$f \equiv F_3(p, q, ay - bx, px + qy - z) = 0, \quad \ldots\ldots\ldots\ldots (3c)$$

where F_1, F_2, F_3 denote arbitrary functions.

Any equation of the form (3a), (3b), (3c), or of similar form constructed as indicated above, is solvable by using the auxiliary equation $ap + bq = c$, where a and b are given constants and c is an arbitrary constant.

Next consider the equation

$$\phi(p, q, ay - bx, az - cx) = 0, \quad \ldots\ldots\ldots\ldots (4)$$

or more generally

$$\psi(f, \phi) = 0, \quad \ldots\ldots\ldots\ldots (5)$$

where f denotes F_1 or F_2 or F_3 (substituting c for $ap + bq$ wherever desired) and ϕ is given by (4). Equations (4) and (5) are not solutions of (1) but are consistent with (1) in virtue of $ap + bq = c$. Hence we get in the general case an ∞^1 of cylinders among the solutions of (4) or (5). But the case of interest is the case where the terms in $ay - bx$ are absent. We then get an ∞^2 of cylinders amongst the solutions. Thus, *the equation*

$$\phi(p, q, az - cx) = 0 \quad \ldots\ldots\ldots\ldots (6)$$

can be solved by using the auxiliary equation $ap + bq = c$, where b is the arbitrary constant.

Another particular case is

$$px + qy - z = \Phi(p, q, z), \quad \ldots\ldots\ldots\ldots (7)$$

which is solvable by putting $p = aq$. The standard form $F(p, q, z) = 0$ is obviously a particular case of (6). C. N. SRINIVASIENGAR.

1402. *The general solution of $ax + by + cz = d$.*

The method of Note 1326 may be easily extended. For example, if x, y, z in $ax + by + cz = d$ are integers, then $(by + cz - d)/a$ is an integer and so is $(kby + kcz - kd)/a$, where $kb \equiv 1 \pmod{a}$.

Let $\quad kb = la + 1, \quad kc = ma + i, \quad kd = na + j.$

Then, substituting, we find $(y + iz - j)/a$ is an integer, say, p—that is,

$$y + iz = j + ap. \quad \ldots\ldots\ldots\ldots (i)$$

Solving (i) in the usual way for $p = 1, 2, 3 \ldots$, we find

$$y = g_p(q), \quad z = h_p(q);$$

and by substitution in the original equation,

$$x = f_p(q).$$

Example. $2x + 9y + 8z = 30,$

$$kb = 9k \equiv 1 \pmod{2} \text{ and so } k = 1.$$

Then

$$kb = 9 = 2l + 1 \text{ and } l = 4,$$

$$kc = 8 = 2m + i,$$

$$kd = 30 = 2n + j.$$

Hence we take $m=3,\quad i=2,$
$n=14,\quad j=2$

and (i) becomes $y+2z=2+2p \quad (p=1, 2, 3, \ldots).$

Using the original notation of Note 1326, here

$$a=1,\quad b=2,\quad c=2(p+1).$$

So $k=1,\quad l=1\ ;$
$kc=2p+2=m+d.$

Hence we take $m=2p+1,\ d=1$

and we find $x=-9p+5q+11,$
$y=2p-2q,$
$z=q+1,$

where $p, q=1, 2, 3, \ldots$. A. F. MACKENZIE.

1403. *A note on coordinate geometry.*

An analytical method of obtaining equations, free from ambiguity, for the in- and ex-centres of a triangle, the equations of the sides of which are given.

Let the equations of the sides, referred to rectangular Cartesian reference axes, be $u_1=0,\ u_2=0,\ u_3=0$, where $u_i \equiv a_i x+b_i y+c_i$, $(i=1, 2, 3)$; and let (x_i, y_i) be the vertex opposite the side $u_i=0$.

By suitably choosing $m_1 : m_2 : m_3$, we can make

$$[m_1u_1+m_2u_2+m_3u_3]$$

independent of x and y and equal to the constant

$$C \equiv m_1c_1+m_2c_2+m_3c_3.$$

For this we require

$$\left.\begin{aligned} m_1a_1+m_2a_2+m_3a_3&=0,\\ m_1b_1+m_2b_2+m_3b_3&=0 \end{aligned}\right\}$$

and hence

$$m_1/(a_2b_3-a_3b_2)=m_2/(a_3b_1-a_1b_3)=m_3/(a_1b_2-a_2b_1).$$

Adopting such a set of values for the m's, and substituting (x_1, y_1) for (x, y) in $m_1u_1+m_2u_2+m_3u_3$, we have, since (x_1, y_1) lies on $u_2=0$ and $u_3=0$,

$$m_1(a_1x_1+b_1y_1+c_1)=C.$$

Similarly, $m_2(a_2x_2+b_2y_2+c_2)=C$

and $m_3(a_3x_3+b_3y_3+c_3)=C.$

Let u_i, after multiplication by the sign of m_i, be denoted by $\alpha_i x+\beta_i y+\gamma_i$. Then the signs of $\alpha_1x_1+\beta_1y_1+\gamma_1$, $\alpha_2x_2+\beta_2y_2+\gamma_2$ and $\alpha_3x_3+\beta_3y_3+\gamma_3$ are, in each case, the same as that of C and so are alike.

Let (x, y) be the in-centre. Then, since (x, y) and (x_i, y_i) are on the same side of $\alpha_i x+\beta_i y+\gamma_i=0$,

$$\frac{\alpha_1x+\beta_1y+\gamma_1}{\alpha_1x_1+\beta_1y_1+\gamma_1},\quad \frac{\alpha_2x+\beta_2+\gamma_2}{\alpha_2x_2+\beta_2y_2+\gamma_2} \quad \text{and} \quad \frac{\alpha_3x+\beta_3y+\gamma_3}{\alpha_3x_3+\beta_3y_3+\gamma_3}$$

are all positive, and since the signs of the denominators are alike, so also must be the signs of the numerators.

Thus $\dfrac{\alpha_1x+\beta_1y+\gamma_1}{\sqrt{[\alpha_1^2+\beta_1^2]}}$, $\dfrac{\alpha_2x+\beta_2y+\gamma_2}{\sqrt{[\alpha_2^2+\beta_2^2]}}$ and $\dfrac{\alpha_3x+\beta_3y+\gamma_3}{\sqrt{[\alpha_3^2+\beta_3^2]}}$

must have the same sign ; also their absolute values must be equal, since these are the lengths of the perpendiculars from (x, y) on the sides.

Hence the coordinates of the in-centre are given by

$$\frac{\alpha_1x+\beta_1y+\gamma_1}{\sqrt{[\alpha_1^2+\beta_1^2]}}=\frac{\alpha_2x+\beta_2y+\gamma_2}{\sqrt{[\alpha_2^2+\beta_2^2]}}=\frac{\alpha_3x+\beta_3y+\gamma_3}{\sqrt{[\alpha_3^2+\beta_3^2]}}.$$

Again, if (x, y) denote the centre of the ex-circle lying within the angle opposite $u_1=0$, then

$$-\frac{\alpha_1x+\beta_1y+\gamma_1}{\sqrt{[\alpha_1^2+\beta_1^2]}}, \quad \frac{\alpha_2x+\beta_2y+\gamma_2}{\sqrt{[\alpha_2^2+\beta_2^2]}} \quad \text{and} \quad \frac{\alpha_3x+\beta_3y+\gamma_3}{\sqrt{[\alpha_3^2+\beta_3^2]}}$$

have the same sign and the same absolute value.

Hence the coordinates of the specified ex-centre are given by

$$-\frac{\alpha_1x+\beta_1y+\gamma_1}{\sqrt{[\alpha_1^2+\beta_1^2]}}=\frac{\alpha_2x+\beta_2y+\gamma_2}{\sqrt{[\alpha_2^2+\beta_2^2]}}=\frac{\alpha_3x+\beta_3y+\gamma_3}{\sqrt{[\alpha_3^2+\beta_3^2]}}.$$

Note.—It is suggested that, when applying these results, the equations of the sides of the triangle, after multiplication by the signs of m_1, m_2 and m_3 respectively, be referred to as the " prepared " equations.

J. H. SMITH.

1404. *The in-centre and ex-centres of a triangle.*

The sides BC, CA, AB of a triangle are given by the equations

$$u\equiv a_1x+b_1y+c_1=0, \quad v\equiv a_2x+b_2y+c_2=0, \quad w\equiv a_3x+b_3y+c_3=0$$

respectively.

To find the coordinates of the in-centre and the ex-centres.

Let $A\equiv(x_1, y_1)$, $B\equiv(x_2, y_2)$, $C\equiv(x_3, y_3)$; and let $I\equiv(x, y)$ be the in-centre.

Let u_r, v_r, w_r denote the values of u, v, w respectively, when x_r, y_r are substituted for x, y respectively.

Then since A and I are on the same side of BC, the expressions u and u_1 have the same sign.

Similarly the expressions v and v_2 have the same sign, and the expressions w and w_3 have the same sign.

Hence if the signs of u_1, v_2, w_3 be denoted by (s_1), (s_2), (s_3) respectively, the coordinates of I are given by the equations

$$(s_1)\frac{u}{\sqrt{(a_1^2+b_1^2)}}=(s_2)\frac{v}{\sqrt{(a_2^2+b_2^2)}}=(s_3)\frac{w}{\sqrt{(a_3^2+b_3^2)}}.$$

Again, since $v_1=0$, $w_1=0$, we have

$$(a_2b_3-a_3b_2)u_1=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\Delta \text{ (say).}$$

Similarly,

$$(a_3b_1-a_1b_3)v_2=\Delta$$

and

$$(a_1b_2-a_2b_1)w_3=\Delta.$$

Hence the equations for the in-centre can be written

$$(\sigma_1)\frac{u}{\sqrt{(a_1^2+b_1^2)}}=(\sigma_2)\frac{v}{\sqrt{(a_2^2+b_2^2)}}=(\sigma_3)\frac{w}{\sqrt{(a_3^2+b_3^2)}}, \quad \text{.....(A)}$$

where σ_1, σ_2, σ_3 are the signs of $a_2b_3-a_3b_2$, $a_3b_1-a_1b_3$, $a_1b_2-a_2b_1$ respectively. We are thus able to write down equation (A) without finding the coordinates of the vertices of the triangle.

The coordinates of the ex-centres I_1, I_2, I_3 of the circles touching BC, CA, AB respectively externally will be given by the equations

$$-(\sigma_1)\frac{u}{\sqrt{(a_1^2+b_1^2)}}=(\sigma_2)\frac{v}{\sqrt{(a_2^2+b_2^2)}}=(\sigma_3)\frac{w}{\sqrt{(a_3^2+b_3^2)}},$$

$$(\sigma_1)\frac{u}{\sqrt{(a_1^2+b_1^2)}}=-(\sigma_2)\frac{v}{\sqrt{(a_2^2+b_2^2)}}=(\sigma_3)\frac{w}{\sqrt{(a_3^2+b_3^2)}},$$

$$(\sigma_1)\frac{u}{\sqrt{(a_1^2+b_1^2)}}=(\sigma_2)\frac{v}{\sqrt{(a_2^2+b_2^2)}}=-(\sigma_3)\frac{w}{\sqrt{(a_3^2+b_3^2)}}.$$

Let us take a numerical example.

Let

$$u\equiv 10x+11y-600,$$

$$v\equiv 22x-71y-3000,$$

$$w\equiv 62x-41y+3000.$$

Here, $a_2b_3-a_3b_2>0$, $a_3b_1-a_1b_3>0$, $a_1b_2-a_2b_1<0$.

Hence I is given by $u:v:w=\sqrt{221}:5\sqrt{221}:-5\sqrt{221}$.

Hence

$$I\equiv(0, 0).$$

I_1 is given by $u:v:w=-1:5:-5$.

F. H. V. GULASEKHARAM.

1405. *A note on the analytical geometry of the straight line.*

1. To find the cosine of the angle between the straight lines

$$u\equiv l_1x+m_1y+n_1=0, \quad v\equiv l_2x+m_2y+n_2=0,$$

within which the point $P_0(x_0, y_0)$ lies.

Transfer the origin to the point P_0. Then the equations of the straight lines become

$$l_1x+m_1y+u_0=0$$

and

$$l_2x+m_2y+v_0=0,$$

where

$$u_0=l_1x_0+m_1y_0+n_1, \quad v_0=l_2x_0+m_2y_0+n_2.$$

Let θ, ϕ be the angles which the perpendiculars $\overrightarrow{P_0N_1}$, $\overrightarrow{P_0N_2}$ *from* P_0 *to* these straight lines make with the *positive* direction of the x-axis.

Then

$$\frac{\cos\theta}{-l_1}=\frac{\sin\theta}{-m_1}=\frac{\mu_1}{\sqrt{(l_1^2+m_1^2)}}$$

and

$$\frac{\cos\phi}{-l_2}=\frac{\sin\phi}{-m_2}=\frac{\mu_2}{\sqrt{(l_2^2+m_2^2)}},$$

where

$$\mu_1=\pm 1 \quad \text{according as} \quad u_0 \gtrless 0$$

and

$$\mu_2=\pm 1 \quad \text{according as} \quad v_0 \gtrless 0.$$

Then, since

$$\cos N_1P_0N_2=\cos(\phi-\theta),$$

the cosine of the angle within which P_0 lies is

$$-\cos(\phi-\theta)=\frac{-\mu_1\mu_2(l_1l_2+m_1m_2)}{\sqrt{\{(l_1^2+m_1^2)(l_2^2+m_2^2)\}}}$$

$$=\frac{\mp(l_1l_2+m_1m_2)}{\sqrt{\{(l_1^2+m_1^2)(l_2^2+m_2^2)\}}}$$

according as $u_0v_0 \gtrless 0$.

Deductions:

(i) The angle within which P_0 lies is acute or obtuse according as

$$u_0v_0(l_1l_2+m_1m_2) \lesseqgtr 0.$$

(ii) The triangle formed by the straight lines $u=0$, $v=0$ and $w\equiv l_3x+m_3y+n_3=0$ is acute- or obtuse-angled according as

$$(l_2l_3+m_2m_3)(l_3l_1+m_3m_1)(l_1l_2+m_1m_2) \lesseqgtr 0.$$

For, if P_0 be any point within the triangle, then the expressions $v_0w_0(l_2l_3+m_2m_3)$, $w_0u_0(l_3l_1+m_3m_1)$, $u_0v_0(l_1l_2+m_1m_2)$ are all negative for an acute-angled triangle, and one positive and the other two negative for an obtuse-angled triangle. Hence the triangle is acute- or obtuse-angled according as

$$u_0^2v_0^2w_0^2(l_2l_3+m_2m_3)(l_3l_1+m_3m_1)(l_1l_2+m_1m_2) \lesseqgtr 0.$$

Obviously, the triangle is right-angled if

$$(l_2l_3+m_2m_3)(l_3l_1+m_3m_1)(l_1l_2+m_1m_2)=0.$$

(iii) If $P(x, y)$ lies within the acute angle between the straight lines $u=0$, $v=0$, then

$$uv(l_1l_2+m_1m_2)<0.$$

Hence if P be any point *on the bisector of the acute angle* between the straight lines $u=0$, $v=0$, we have

$$u/\sqrt{\{l_1^2+m_1^2\}}=\pm v/\sqrt{\{l_2^2+m_2^2\}}$$

according as $l_1l_2+m_1m_2 \lesseqgtr 0$, which gives the bisector of the acute angle.

Note.—In numerical examples, the following *extremely simple method* is available. I have not seen it anywhere.

Suppose

$$u \equiv 3x+\ 4y-5,$$
$$v \equiv 5x+12y-3.$$

Then the bisectors of the two angles between $u=0$ and $v=0$ are

$$p \equiv 7x-4y-25=0$$

and

$$q \equiv 4x+7y-\ 5=0.$$

The acute angle between the straight lines $u=0$ and $p=0$ is

$$\tan^{-1}\left|\frac{(10/4)}{1-\frac{21}{16}}\right| = \tan^{-1}(8) > 45°.$$

Thus $p=0$ bisects the obtuse angle between the straight lines $u=0$, $v=0$; and $q=0$ bisects the acute angle.

F. H. V. GULASEKHARAM.

1406. *A question and solution.*

The following problem was proposed in the *Nouvelles Annales* in 1896.

"If $a \equiv \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} \neq 0$, $b \equiv \begin{vmatrix} b_{11}, & b_{12}, & b_{13} \\ b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \end{vmatrix} \neq 0$,

$$d \equiv \begin{vmatrix} a_{12}b_{13}-a_{13}b_{12}, & a_{22}b_{23}-a_{23}b_{22}, & a_{32}b_{33}-a_{33}b_{32} \\ a_{13}b_{11}-a_{11}b_{13}, & a_{23}b_{21}-a_{21}b_{23}, & a_{33}b_{31}-a_{31}b_{33} \\ a_{11}b_{12}-a_{12}b_{11}, & a_{21}b_{22}-a_{22}b_{21}, & a_{31}b_{32}-a_{32}b_{31} \end{vmatrix}$$

and

$$D \equiv \begin{vmatrix} A_{12}B_{13}-A_{13}B_{12}, & A_{22}B_{23}-A_{23}B_{22}, & A_{32}B_{33}-A_{33}B_{32} \\ A_{13}B_{11}-A_{11}B_{13}, & A_{23}B_{21}-A_{21}B_{23}, & A_{33}B_{31}-A_{31}B_{33} \\ A_{11}B_{12}-A_{12}B_{11}, & A_{21}B_{22}-A_{22}B_{21}, & A_{31}B_{32}-A_{32}B_{31} \end{vmatrix},$$

where B_{ik} is the minor of b_{ik}, A_{ik} the minor of a_{ik}, then $D=abd$."

Solution. Let

$$a_{11}x+a_{12}y+a_{13}z=0$$

be the equation to the straight line A_2A_3 (see fig., p. 390) referred to an arbitrary system of trilinear coordinates, and

$$b_{11}x+b_{12}y+b_{13}z=0$$

the equation of B_2B_3 referred to the same system. Then the coordinates of D_1 are $(a_{12}b_{13}-a_{13}b_{12}, a_{13}b_{11}-a_{11}b_{13}, a_{11}b_{12}-a_{12}b_{11})$ and the coordinates of the straight line A_1B_1 are $(A_{12}B_{13}-A_{13}B_{12}, \ldots, \ldots)$.

If instead of choosing an arbitrary triangle of reference we take $B_1B_2B_3$, then $b_{ik}=0$ $(i\neq k)$ and

b becomes $\begin{vmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{vmatrix}$;

d becomes $\begin{vmatrix} 0 & -a_{23}b_{22} & a_{32}b_{33} \\ a_{13}b_{11} & 0 & -a_{31}b_{33} \\ -a_{12}b_{11} & a_{21}b_{22} & 0 \end{vmatrix}$.

Further, $B_{ik}=0 \quad (i\neq k)$

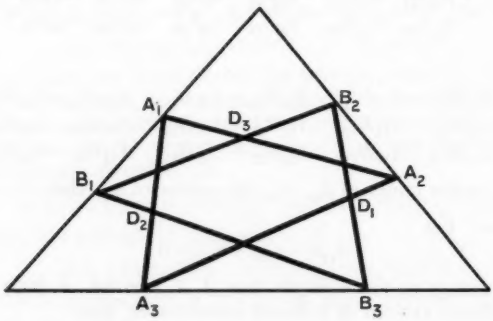
and $B_{11}=b_{22}b_{33}, \quad B_{22}=b_{33}b_{11}, \quad B_{33}=b_{11}b_{22}.$

Thus $D=\begin{vmatrix} 0 & -A_{23}B_{22} & A_{32}B_{33} \\ A_{13}B_{11} & 0 & -A_{31}B_{33} \\ -A_{12}B_{11} & A_{21}B_{22} & 0 \end{vmatrix}$.

The equation $D=abd$ becomes

$$B_{11}B_{22}B_{33}\begin{vmatrix} 0 & -A_{23} & A_{32} \\ A_{13} & 0 & -A_{31} \\ -A_{12} & A_{21} & 0 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot (b_{11}b_{22}b_{33})^2 \cdot \begin{vmatrix} 0 & -a_{23} & a_{32} \\ a_{13} & 0 & -a_{31} \\ -a_{12} & a_{21} & 0 \end{vmatrix}$$

or $$\begin{vmatrix} 0 & -A_{23} & A_{32} \\ A_{13} & 0 & -A_{31} \\ -A_{12} & A_{21} & 0 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} 0 & -a_{23} & a_{32} \\ a_{13} & 0 & -a_{31} \\ -a_{12} & a_{21} & 0 \end{vmatrix}.$$



Now the right-hand product is equal to

$$\begin{vmatrix} 0 & A_{32} & -A_{23} \\ -A_{31} & 0 & A_{13} \\ A_{21} & -A_{12} & 0 \end{vmatrix}.$$

which is equal to the left-hand side, and so the assertion is justified.

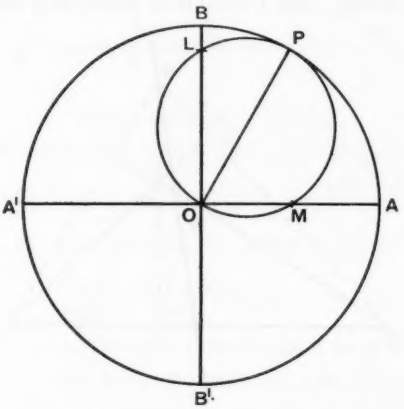
If $d=0$, then $D=0$, since $a\neq 0$, $b\neq 0$;

if $D=0$, then $d=0$, since $a\neq 0$, $b\neq 0$.

This is Desargues' theorem. D. ARANY.

1407. *A trigonometrical device.*

The device illustrated in the figure is one I use in demonstrating the variation of sin x and cos x. It consists of two plywood circles



(in my instrument the radii are 5 inches and 10 inches), the smaller passing through the centre O of the larger. If OP be a diameter of the smaller circle and AOA', BOB' two perpendicular diameters of the larger graduated from -1 to $+1$, the intersections of the smaller circle with AOA', BOB' of course record cos x and sin x when x is the angle POA. The visual aid seems to be useful with some boys.

A quite cheap thin plywood can be used, and prepared for drawing with Indian ink by painting in white undercoating very much thinned with turpentine. Incidentally this, well glass-papered, is a useful material for drawings which are to have a more or less permanent existence in the classroom.

A. G. GRANSTON RICHARDS.

1408. *Why do teachers always draw acute-angled triangles?*

If anyone will take the trouble to draw five or six lines at random and count the number of triangles formed which are acute-angled and obtuse-angled respectively, he or she will easily be convinced that the obtuse-angled triangles are in a large majority.

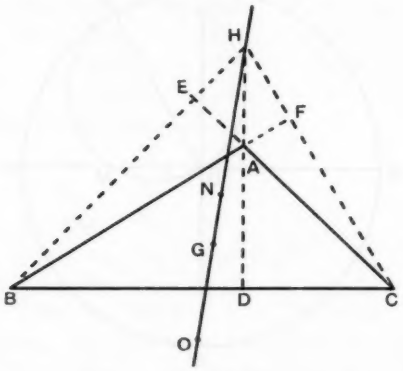
What the chance of a triangle drawn " at random " being obtuse-angled may be depends on the interpretation given to the words " at random ". I have tried various methods and get answers

varying from ·57 to ·75, the latter being got very simply by two methods :

(i) considering that the greatest angle must lie between 60° and 180° ;

(ii) taking a large number of points evenly spread round a circle and counting the triangles formed by joining the points in every possible way.

Anyhow, as obtuse-angled triangles are at least the more probable, the usual figure of a triangle with its circumcentre and orthocentre should be as shown. But I venture to assert that this figure is an



unfamiliar one to the large majority of pupils and possibly even to a minority of teachers. How many who read this are quite sure whether or not the nine-point centre is necessarily *inside* the original triangle?

The moral is : Draw more obtuse-angled triangles ; or if a shorter slogan be desired : " Be obtuse-minded ". C. O. TUCKEY.

1409. *The addition formulae.*

The following proofs are submitted as being suitable for technical students, since they depend only upon the easily demonstrated formulae $\Delta = \frac{1}{2}ab \sin C$ and the associated

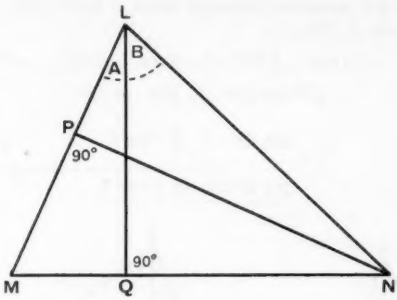
$$a/\sin A = b/\sin B = c/\sin C = 2\Delta/abc.$$

Many books use the former for proving the formulae for $\sin (A \pm B)$ but a totally different method for $\cos (A \pm B)$.

1. To prove that

$$\cos \overline{A+B} = \cos A \cos B - \sin A \sin B.$$

With the diagram for proving the formula for $\sin \overline{A+B}$ draw NP perpendicular to LM.



We have

$$\triangle PNL = \triangle MNL - \triangle MNP,$$

or $\frac{1}{2}PN \,.\, LN \sin\{\frac{1}{2}\pi - (A+B)\} = \frac{1}{2}MN \,.\, LN \sin(\frac{1}{2}\pi - B) - \frac{1}{2}MN \,.\, PN \sin A$;

that is,

$$\begin{aligned}\cos(A+B) &= (MN/PN)\cos B - (MN/LN)\sin A \\ &= \frac{\cos B}{\cos A} - \frac{\sin(A+B)}{\cos A}\sin A \\ &= (\cos B - \sin^2 A \cos B - \cos A \sin A \sin B)/\cos A \\ &= (\cos B \cos^2 A - \cos A \sin A \sin B)/\cos A \\ &= \cos A \cos B - \sin A \sin B.\end{aligned}$$

2. To prove that

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

In the above diagram let $\angle MLN = A$, $\angle MLQ = B$, so that $\angle QLN = A - B$. Then

$$\triangle LNM = \triangle PNL + \triangle PNM.$$

Thus

$$\frac{1}{2}MN \,.\, LN \,.\, \sin\{\frac{1}{2}\pi - (A-B)\} = \frac{1}{2}PN \,.\, LN \sin(\frac{1}{2}\pi - A) + \frac{1}{2}MN \,.\, PN \sin B;$$

that is,

$$\begin{aligned}\cos(A-B) &= (PN/MN)\cos A + (PN/LN)\sin B \\ &= \cos A \cos B + \sin A \sin B.\end{aligned}$$

J. PEDOE.

1410. *Some trigonometrical proofs.*

The following proofs might be of some interest and value in teaching. As will be seen, they are based on the elementary formula for the area of a triangle, viz. area $= \frac{1}{2}ab \,.\, \sin C$.

Let ABC be an isosceles triangle with $\angle BAC=2\theta$. Let AD be the altitude from A (Fig. 1).

$$\text{Area } \triangle ABC=2 \text{ . Area } \triangle ABD.$$

Thus $$\tfrac{1}{2}b^2 \sin 2\theta=2 \text{ . } \tfrac{1}{2}bp \text{ . } \sin\theta$$

or $$\sin 2\theta=2 \cdot \frac{p}{b} \cdot \sin\theta$$

and $$\underline{\sin 2\theta=2 \sin\theta \cos\theta.}$$

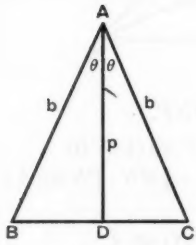


FIG. 1.

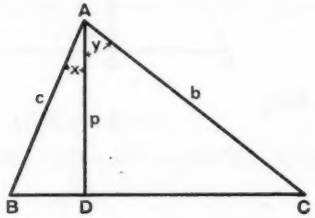


FIG. 2.

Let ABC be any triangle and AD the altitude such that

$$\angle BAD=X \text{ and } \angle DAC=Y,$$

so that $\angle BAC=(X+Y)$ (Fig. 2).

$$\text{Area } \triangle ABC=\text{Area } \triangle ABD+\text{Area } \triangle ADC.$$

Thus $$\tfrac{1}{2}bc \text{ . } \sin(X+Y)=\tfrac{1}{2}pc \text{ . } \sin X+\tfrac{1}{2}bp \text{ . } \sin Y$$

or $$\sin(X+Y)=\frac{p}{b} \cdot \sin X+\frac{p}{c} \text{ . } \sin Y$$

and $$\underline{\sin(X+Y)=\sin X \cos Y+\sin Y \cos X.}$$

Now let $\angle ABC$ be obtuse, so that $\angle BAC=(X-Y)$ (Fig. 3).

$$\text{Area } \triangle ABC=\text{Area } \triangle ADC-\text{Area } \triangle ABD.$$

Thus $$\tfrac{1}{2}bc \text{ . } \sin(X-Y)=\tfrac{1}{2}bp \text{ . } \sin X-\tfrac{1}{2}cp \text{ . } \sin Y$$

or $$\sin(X-Y)=\frac{p}{c} \text{ . } \sin X-\frac{p}{b} \text{ . } \sin Y$$

and $$\underline{\sin(X-Y)=\sin X \cos Y-\sin Y \cos X.}$$

For the corresponding cosine formulae the above figures, with the addition of the altitude from C, can be used in the following manner:

Let AD and EC intersect at H and let $CH=d$ and $CD=h$ (Fig. 4).

From the figure, it is obvious that

$$\angle HCD=\theta, \ \angle ECA=90^\circ-2\theta \text{ and } \angle ACD=90^\circ-\theta.$$

$$\text{Area } \triangle AHC=\text{Area } \triangle ADC-\text{Area } \triangle HCD.$$

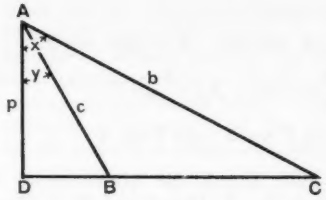


FIG. 3.

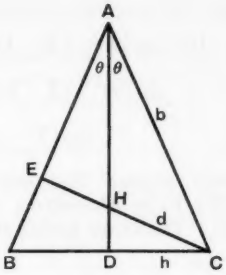


FIG. 4.

Thus

$$\tfrac{1}{2}bd \,.\, \sin(90° - 2\theta) = \tfrac{1}{2}hb \,.\, \sin(90° - \theta) - \tfrac{1}{2}dh \,.\, \sin\theta$$

or

$$\sin(90° - 2\theta) = \frac{h}{d} \cdot \sin(90° - \theta) - \frac{h}{b} \cdot \sin\theta$$

and

$$\underline{\cos 2\theta = \cos^2\theta - \sin^2\theta.}$$

As before, let $CH = d$, $CD = h$. From Fig. 5, $\angle HCD = X$, $\angle ECA = 90° - (X + Y)$ and $\angle ACD = 90° - Y$.

$$\text{Area } \triangle AHC = \text{Area } \triangle ADC - \text{Area } \triangle HCD.$$

Thus

$$\tfrac{1}{2}bd \,.\, \sin[90° - (X + Y)] = \tfrac{1}{2}hb \,.\, \sin(90° - Y) - \tfrac{1}{2}dh \,.\, \sin X$$

or

$$\sin[90° - (X + Y) = \frac{h}{d} \cdot \sin(90° - Y) - \frac{h}{b} \cdot \sin X$$

and

$$\underline{\cos(X + Y) = \cos X \cos Y - \sin Y \sin X.}$$

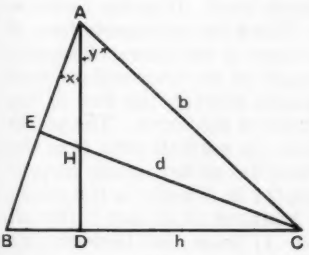


FIG. 5.

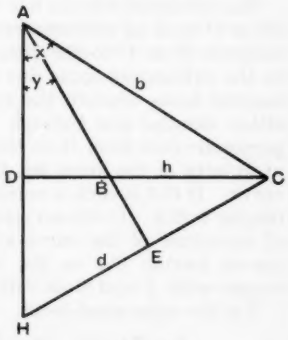


FIG. 6.

Again, let $CH = d$, $CD = h$. From Fig. 6, $\angle HCD = Y$, $\angle ECA = 90° - (X - Y)$ and $\angle ACD = 90° - X$.

$$\text{Area } \triangle AHC = \text{Area } \triangle ADC + \text{Area } \triangle HCD.$$

Thus

$$\tfrac{1}{2}bd \,.\, \sin[90° - (X - Y)] = \tfrac{1}{2}bh \,.\, \sin(90° - X) + \tfrac{1}{2}hd \,.\, \sin Y$$

or
$$\sin[90° - (X - Y)] = \frac{h}{d}\sin(90° - X) + \frac{h}{b}\sin Y$$

and
$$\underline{\cos(X - Y) = \cos Y \cos X + \sin Y \sin X.}$$

Since writing this note I have found the above proof for $\sin(X + Y)$ in *Elementary Analysis*, by A. Dakin and R. I. Porter (Bell). The other proofs are however not given.

E. E. I.

1411. *On the loci of the ends of polar subtangent and polar subnormal.*

If P is any point of a curve, the intersections of the line through the pole O perpendicular to the radius vector OP with the tangent and normal at P are called respectively the ends of the polar subtangent and polar subnormal of P. The loci of these two ends, which will be called the "subtangent-locus" and "subnormal-locus" respectively, have some interesting properties which are perhaps not well known and it seems worth while to put them on record.

Suppose that the degree, class, number of nodes, of cusps, of inflexions, of bitangents of the given curve (assumed algebraic) are denoted as usual by n, m, δ, κ, ι, τ and its deficiency by D. Let the same quantities for the subtangent-locus be denoted by $n_1, m_1, \ldots$, and for the subnormal-locus by $n_2, m_2, \ldots$. Then in general

$$n_1 = n + m, \quad m_1 = 3m + \kappa, \quad \kappa_1 = 0, \quad D_1 = D.$$

The subtangent-locus has an m-ple point (multiple point of order m) at O with an inflexion on each branch at O. It is due to the m tangents from O to the given curve. There are κ tangents from O to the subtangent-locus due to the κ cusps of the curve. The subtangent-locus touches the curve at each of its intersections with either circular line through O and passes through the feet of the perpendiculars from O on the asymptotes of the curve. The points at infinity of the locus are due to the $n+m$ normals from O to the curve. If OA is such a normal, the locus has an asymptote perpendicular to OA. If this asymptote meets OA in N and C is the centre of curvature of the curve at A, the positions of N and C (for all curves having OA as the normal at A) trace out homographic ranges with A and O as vanishing points respectively.

For the subnormal-locus

$$n_2 = 2n + m, \quad m_2 = 2n + 3m + \kappa, \quad \kappa_2 = 0, \quad D_2 = D.$$

The locus has O as an $(n+m)$-ple point. If OA is the normal at A to the curve and C is the centre of curvature at A, CO is a normal at O to a branch of the locus; and if C' is its centre of curvature, the positions of C and C' (for all curves having OA as the normal at

A) trace out homographic ranges, whose self-corresponding points are O and the mid-point of OA.

The line at infinity touches the subnormal-locus at n points due to the n points at infinity on the given curve, the latus rectum of each osculating parabola being equal to the distance from O of the corresponding asymptote of the curve. The line at infinity also meets the locus at m points due to the tangents from O to the curve. The asymptotes touching the locus at these m points pass through O. The only other tangents from O to the locus are those due to the cusps of the given curve. The locus passes through the intersections of the curve with the circular lines through O.

These results will require modification, if the given curve is specialised in any manner. As space is limited, a few examples must suffice. If O lies on the curve, the subtangent-locus has a cusp at O. If O is a cusp of the curve, the subtangent-locus has a superlinear branch of order 3 and the subnormal-locus an ordinary branch touching the normal to the curve at O. If the curve passes through the circular points at infinity I and J, the two loci both touch the curve at I and J whether the curve touches the line at infinity at I and J or not. If the curve has cusps at I and J, both loci have cusps at I and J with the same tangents as the curve (a rhamphoid cusp for the subnormal-locus if IJ is not the tangent at I and J).

HAROLD SIMPSON.

1412. *An elementary derivation of the relation between trigonometric and hyperbolic functions.*

During the presentation of differential properties of the trigonometric and hyperbolic functions to elementary students, it is useful to have available the imaginary relations between the two sets of functions. The following derivation of this, based altogether on familiar results, has been found quite satisfactory. It avoids any artificial introduction of complex numbers or any recourse to infinite series.

Let $$y = \cos^{-1}\cosh x,$$

then

$$\frac{dy}{dx} = \frac{\pm 1}{\sqrt{(1-\cosh^2 x)}}\,\frac{d}{dx}\cosh x$$

$$= \frac{\pm \sinh x}{\sqrt{(1-\cosh^2 x)}}$$

but $$1 - \cosh^2 x = -\sinh^2 x;$$

hence $$\frac{dy}{dx} = \frac{\pm \sinh x}{\sqrt{(-\sinh^2 x)}} = \frac{\pm 1}{\sqrt{(-1)}}$$

$$= \mp i.$$

From this, $$dy = \mp i\,dx$$

or $$y = \mp ix + K.$$

That is, $$\mp ix + K = \cos^{-1} \cosh x$$

or $$\cosh x = \cos(\mp ix + K)$$
$$= \cos(ix \mp K).$$

Setting $x=0$, we may choose $K=0$ so that

$$\cosh x = \cos ix,$$

from which all the other relations follow.

It is essential, of course, to point out at the beginning that the range of definition of the arc cosine has been extended to include values of the independent variable greater than one, the formal properties being retained and the results of the extension being evident in the discussion. NEWMAN A. HALL.

1413. *A theorem in elementary solid geometry.*

This Note gives an extension of the theorem that *the sum of the exterior angles of a convex plane polygon is* 2π.

Since the exterior angle in plane geometry is defined as "π minus the interior angle", let the exterior angle in solid geometry be defined as "2π minus the sum of the face angles of the solid angle". Then *the sum of the exterior angles of a convex polyhedron is* 4π.

Consider any face, having V_n vertices, of a convex polyhedron of F faces, E edges and V vertices. If S_n is the sum of the interior angles of this face, then by the plane theorem

$$S_n = (V_n - 2)\pi.$$

Also, if Σ is the sum of all the face angles of the polyhedron,

$$\begin{aligned}\Sigma &= S_1 + S_2 \ldots + S_F \\ &= \{(V_1 - 2) + (V_2 - 2) + \ldots + (V_F - 2)\}\pi \\ &= \{(V_1 + V_2 \ldots + V_F) - 2F\}\pi \\ &= (2E - 2F)\pi \\ &= (E - F) \,.\, 2\pi \\ &= (V - 2) \,.\, 2\pi,\end{aligned}$$

by Euler's theorem that $V + F = E + 2$.

Thus $$2\pi V - \Sigma = 4\pi.$$

But the sum of the exterior angles of a polyhedron is, by definition,

$$(V \,.\, 2\pi - \Sigma) = 4\pi.$$

I am indebted to Mr. H. Lob for informing me that the theorem appears in Rouché et Comberousse, *Traité de Géométrie*, Vol 2, 109 (1922), in the form "the sum of all the angles of the faces of a polyhedron is $4(S-2)$ right angles, where S is the number of summits". But the simplicity of statement due to the above definition of "exterior angle" may be worth making, in view of the small notice given to this result in English textbooks.

R. H. MACMILLAN.

1414. *The "tails" of the Steinerian.*

If the first polar of a point $P(\xi, \eta, \zeta)$ with respect to the plane curve $f(x, y, z) = 0$ has a double point at $Q(X, Y, Z)$, the locus * of P is the Steinerian of the given curve. If the given curve is of order n, the Steinerian is usually regarded as a curve of order $3(n-2)^2$, its equations being obtained by eliminating X, Y, Z from

$$\xi \frac{\partial^2 f}{\partial X^2} + \eta \frac{\partial^2 f}{\partial X \partial Y} + \zeta \frac{\partial^2 f}{\partial X \partial Z} = 0$$

and two similar equations.

The object of this note is to point out that besides the curve so obtained by elimination, there may often exist a number of straight lines which satisfy the definition of the Steinerian.

Choosing the vertex (1, 0, 0) as a point on the Steinerian and the vertex (0, 0, 1) as the corresponding point on the Hessian, the equation of the given curve can be written

$$f \equiv az^n + bz^{n-1}y + cz^{n-2}y^2 + z^{n-3}(d_0x^3 + 3d_1x^2y + \ldots) + \ldots = 0.$$

The first polar of (x_1, y_1, z_1) is given by

$$z^{n-1}(by_1 + naz_1) + z^{n-2}y(2cy_1 + \overline{n-1}bz_1) + \ldots = 0.$$

Suppose now that the coefficients satisfy the relation

$$(n-1)b^2 = 2nac.$$

Then the line $by + naz = 0$ is identical with the line $2cy + (n-1)bz = 0$, and the first polar of any point on this line has a double point at (0, 0, 1). *The entire line is therefore a part of the Steinerian*, and every point on this line has (0, 0, 1) as its "corresponding point" on the Hessian. A particular case is when $a = b = 0$, in which case (0, 0, 1) is a cusp of the given curve and the above line becomes the cuspidal tangent $y = 0$. *Every cuspidal tangent of the given curve is thus a part of the Steinerian.*

The condition $(n-1)b^2 = 2nac$ *is the condition that the polar conic of* (0, 0, 1) *should break up into two coincident straight lines*; in other words, when this condition is satisfied, the usual equations for determining the Steinerian are equivalent to a single equation.

Attached to the curve proper, the Steinerian according to definition can thus admit a number of *tails* consisting of straight lines each of which when taken twice constitutes the degenerate polar conic of some point. Of course, these lines need not exist for any given curve.

It can also be verified that the condition $(n-1)b^2 = 2nac$ makes the point (0, 0, 1) a double point on the Hessian. A double point on the Hessian, not lying on the given curve, may, however, arise in another way, viz. when $d_0 = d_1 = 0$. The full significance of these conditions requires elucidation. C. N. SRINIVASIENGAR.

* To secure precision, it must be understood that Q is not a triple or higher multiple point on the given curve, for at such a point the first polar of every point in the plane will have a multiple point.

1415. *Series solution of differential equations.*

There is a point concerning the series solution of linear differential equations which seems to have escaped the writers of at least the more elementary texts.

As an illustrative example, the substitution

$$y = a_0x^n + a_1x^{n+1} + a_2x^{n+2} + \dots$$

in the equation

$$d^2y/dx^2 = x^{10}y$$

would lead to a large majority of vanishing coefficients, since the sequences of indices in the two fundamental solutions are $(0, 12, 24, \dots)$ and $(1, 13, 25, \dots)$—or, as we should say, the *index interval* is 12. The index interval can easily be discovered from the differential equation by trial (*e.g.* by noting the indices of the contribution of a_0x^n). If this is first done, and r is the index interval found, then the trial series

$$y = a_0x^n + a_1x^{n+r} + a_2x^{n+2r} + \dots$$

will lead to *no* vanishing coefficients.

The idea of the index interval also clears up the (apparent) anomaly, referred to in the textbooks, that an integral difference between the roots of the indicial equation does not always lead to trouble in obtaining a second solution (as in the case of Bessel functions of order $n+\frac{1}{2}$). The trouble occurs only when the difference between these roots is a multiple of the index interval.

W. G. B.

1416. *An A.R.P. Problem.*

Mr. Cox in the May *Gazette* estimated the probable value of r, the number of hits when n bombs are dropped at random on an area divided into N units of which R units are vulnerable. I give below an estimate of the standard deviation of this random variable. Before doing so it may be useful to point out that the probable value of r can legitimately be calculated by multiplying $1-\left(\frac{N-1}{N}\right)^n$ by R and that no complicated manipulation is necessary.

If x is a random variable, denote its " expectation " or probable value by $E(x)$. Let x and y be two random variables, whether independent or not, then $E(x+y) = E(x) + E(y)$. Consider a set of n events $A_1, A_2, \dots, A_n$ and a set of random variables $x_1, x_2, \dots, x_n$, so that x_i takes the value ξ_i when A_i occurs and the value 0 when A_i does not occur, then

$$E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} E(x_i) = \sum_{i=1}^{n} \xi_i p(A_i),$$

where $p(A_i)$ is the probability that A_i occurs. If $\xi_i = 1$ for each i, $E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} p(A_i)$, and since now $\sum_{i=1}^{n} x_i = r$ is the number of events which occur, $E(r) = \Sigma p(A_i)$. It is also easy to prove that

$E(r^2) = \sum_{i=1}^{n} p(A_i) + 2\sum_{i,j} p(A_i A_j)$, where $p(A_i A_j)$ is the probability that both A_i and A_j occur. More generally it can be shown that $E(r^k) = \sum_{\mu=1}^{k} S_\mu \Delta^\mu(0^k)$, where $S_\mu = \Sigma p(A_{\alpha_1} A_{\alpha_2} \dots A_{\alpha_\mu})$ and the extent of the summation is defined by $1 \leqslant \alpha_1 < \alpha_2 \dots < \alpha_\mu \leqslant n$.

In the A.R.P. problem the probability that a specified target is hit is $1 - \left(\frac{N-1}{N}\right)^n$, and since there are R such targets,

$$E(r) = R\left\{1 - \left(\frac{N-1}{N}\right)^n\right\}.$$

The probability that two specified targets are hit is

$$1 - 2\left(\frac{N-1}{N}\right)^n + \left(\frac{N-2}{N}\right)^n$$

and there are $\binom{R}{2}$ pairs of targets, hence from the formula for $E(r^2)$,

$$E(r^2) = R\left\{1 - \left(\frac{N-1}{N}\right)^n\right\} + R(R-1)\left\{1 - 2\left(\frac{N-1}{N}\right)^n + \left(\frac{N-2}{N}\right)^n\right\}.$$

Therefore the square of the standard deviation, which is denoted by $\sigma^2(r)$ and equals $E(r^2) - (E(r))^2$, is given by

$$\sigma^2(r) = R\left\{\left(\frac{N-1}{N}\right)^n - \left(\frac{N-2}{N}\right)^n\right\} + R^2\left\{\left(\frac{N-2}{N}\right)^n - \left(\frac{N-2+1/N}{N}\right)^n\right\}.$$

The term in R^2 is clearly negative and the term in R is less than $Re^{-n/N}$. It can also be shown that it is less than $Re^{-1+1/N}$.

A knowledge of $E(r)$ and of $\sigma(r)$ helps to some extent to gauge the probabilities arising out of this problem. A theorem due to Tchebycheff asserts that the probability that $|r - E(r)| \geqslant \lambda\sigma(r)$ does not exceed $\frac{1}{\lambda^2}$.

In general the $\frac{1}{\lambda^2}$ of Tchebycheff's theorem is a gross over-estimate of the actual probability, but there are cases when it gives useful results.

The discussion of the problem falls naturally into three cases: (i) when $\frac{n}{N}$ is large, (ii) when $\frac{n}{N}$ is small, and (iii) when $\frac{n}{N}$ is moderate. It is assumed in each case that N is not small.

In (i) $E\left(\frac{r}{R}\right) \simeq 1$, $\sigma^2\left(\frac{r}{R}\right) \simeq R^{-1}e^{-n/N}$.

In (ii) $E\left(\frac{r}{R}\right) \simeq \frac{n}{N}$, $\sigma^2\left(\frac{r}{R}\right) \simeq \frac{n}{N}\Big/R$.

In (iii) $E\left(\frac{r}{R}\right) \doteq 1 - e^{-n/N}$, and an upper-estimate of $\sigma^2\left(\frac{r}{R}\right)$ is given by

$$\sigma^2\left(\frac{r}{R}\right) \leqslant \min\{R^{-1}e^{-1+1/N},\ R^{-1}e^{-n/N}\}.$$

As an illustration of the information yielded by Tchebycheff's theorem, suppose $n = N$ and let $R = 3600$.

$$E\left(\frac{r}{R}\right) \doteq 1 - e^{-1} = \cdot 23\ldots,\quad \sigma\left(\frac{r}{R}\right) \leqslant R^{-\frac{1}{2}}\, e^{-\frac{1}{2}} = \frac{1}{60e^{\frac{1}{2}}}.$$

Thus the probability that $\left|\frac{r}{R} - \cdot 63\right| > \frac{1}{10}$ is less than $\frac{1}{36e}$ which is roughly $\frac{1}{98}$.

For a more accurate gauging of probabilities use could be made of the methods of Gram-Charlier and Pearson for approximating to the probability distribution. The formula for $E(r^k)$ given above will supply the " moments " needed for this process.

T. S. Broderick.

1417. *The triple vector product.*

With the usual tensor notation

$$(b \wedge c)_i = \epsilon_{ikl} b^k c^l,\quad (b \wedge c)^i = \epsilon^{ikl} b_k c_l.$$

$$\begin{aligned}\{a \wedge (b \wedge c)\}^j &= \epsilon^{jmi} a_m (b \wedge c)_i \\ &= \epsilon^{jmi}\epsilon_{ikl} a_m b^k c^l \\ &= \delta^{jm}_{kl} a_m b^k c^l \\ &= a_m (b^j c^m - b^m c^j) \\ &= (a_m c^m) b^j - (a_m b^m) c^j;\end{aligned}$$

that is $$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

F. M. Saxelby.

1418. *On the identity discussed in Notes* 1310, 1311, 1348, 1349, 1377.

The identity under discussion is closely connected with, and easily derivable from, one which presented itself very early in the history of the theory of probabilities, in the form of two different expressions for the same chance. De Montfort in effect showed, in his discussion of the Problem of Points,* that if p and q are the respective chances of success and failure in a single trial and if for brevity

* *Essai d'Analyse sur les Jeux de Hazards*, 2nd ed. (1714), pp. 232-48; see (1) Todhunter's *History of the Theory of Probability*, pp. 97-8. Cf. (2) Todhunter's *Algebra*, p. 458, (3) Chrystal's *Algebra*, Part II, p. 583. Different algebraic proofs of De Montfort's identity are given in (1), (2) and (3). The opportunity may be taken to point out that Todhunter's indispensable *History*, so long out of print and practically unobtainable, has been reprinted in the U.S.A. (New York, G. E. Stechert & Co., 1931).

we write c_s for $\binom{r+s-1}{s}$, the chance that the event will happen at least r times in m trials may be represented either by the expression

$$p^r[1 + c_1q + c_2q^2 + \ldots + c_{m-r}q^{m-r}]$$

or, as usual, by the sum of the first $m-r+1$ terms in the expansion of $(p+q)^m$. Putting $m = 2n-1$, $r = n$, and so $c_s \equiv \binom{n+s-1}{s}$, we thus get

$$p^n[1 + c_1q + \ldots + c_{n-1}q^{n-1}] = \text{sum of 1st } n \text{ terms of } (p+q)^{2n-1}.$$

From this, by interchanging p and q, we get

$$q^n[1 + c_1p + \ldots + c_{n-1}p^{n-1}] = \text{sum of last } n \text{ terms of } (p+q)^{2n-1}.$$

Therefore, since the whole expansion of $(p+q)^{2n-1}$ equals unity, we get

$$p^n[1 + c_1q + \ldots + c_{n-1}q^{n-1}]$$
$$+ q^n[1 + c_1p + \ldots + c_{n-1}p^{n-1}] = 1,$$

which is a form of the identity under discussion. To recover the usual form, as given in the Notes, we have only to multiply both sides by $(\alpha+\beta)^n$ and put $p = \alpha/(\alpha+\beta)$, $q = \beta/(\alpha+\beta)$: cf. Stokes' proof, Note 1377.

The above proof shows, what does not appear from the other proofs given, that the identity is the sum of two symmetrically related identities of a rather complicated form.

The following is a simple and straightforward algebraic proof of de Montfort's important identity. Using the recognized and compact notation

$$\binom{n}{s} \equiv n_{(s)}, \text{ so that } \left[\begin{matrix} n \\ s \end{matrix}\right] \equiv (n+s-1)_{(s)}$$

the identity is, if $p+q=1$,

(1) $$p^n + n_{(1)}p^{n-1}q + n_{(2)}p^{n-2}q^2 + \ldots + n_{(n-r)}p^rq^{n-r}$$
$$=$$
(2) $$p^r[1 + r_{(1)}q + (r+1)_{(2)}q^2 + \ldots + (n-1)_{(n-r)}q^{n-r}].$$

Putting $p = 1-q$, (1) may be written as

$$p^r[(1-q)^{n-r} + n_{(1)}(1-q)^{n-r-1}q + \ldots + n_{(n-r)}q^{n-r}].$$

Expanding the powers of $(1-q)$ and collecting coefficients of q^t we get

From 1st term $\quad (-)^t(n-r)_{(t)} \cdot 1$

„ 2nd term $\quad (-)^{t-1}(n-r-1)_{(t-1)} \cdot n_{(1)}$

„ 3rd term $\quad (-)^{t-2}(n-r-2)_{(t-2)} \cdot n_{(2)}$

.....

„ $(t+1)$th term $\quad 1 \cdot n_{(t)}$.

Considering these in reverse order it is seen that their sum is the coefficient of x^t in the expansion of

$$(1+x)^n \cdot (1+x)^{-(n-r-t+1)}$$

i.e. in $(1+x)^{(r+t-1)}$

i.e. $(r+t-1)_{(t)}$

and this establishes the identity.

In view of the widely different proofs given in the Notes it would be interesting to know how, where and by whom the identity was first established. G. J. LIDSTONE.

[Dr. Hayden and Mr. Andress say that they met the identity in question in connection with the problem of finding an interpolation formula for the symmetric part of a function.—Ed.]

CORRESPONDENCE.

SIR,—I have always felt a little unhappy about teaching children that an unknown x is a number, because in this case it is difficult to argue that the equation

$$s = u + \frac{f}{t^2}$$

cannot be right because of the "dimensions". A third-former, however (who has, I trust, received no indication of my unhappiness) took it upon himself, when in a recent question he was finding the weight of a cartload and a lorry load, to begin

"Let a cartload $= x$,
a lorry load $= y$,
and one ton $= t$",

finally obtaining $x = 1\frac{3}{5}t = 1$ ton 12 cwt.

This seems to me a neat way out of my difficulty and only with great regret did I dock his marks.

Yours faithfully, CHARLES HAWES.

1289. We find to our Surprise, that there are several *Mathematicians* and professed *Well-wishers* to Science, who reproach the *Backwardness* of others to promote laudable and useful Science, who, though they can afford it, will not themselves be at a small Expence in purchasing the Books of Science published they approve; but use other Means to inspect them.—Who, not setting Example in what they so earnestly recommend, are like those pious *Inveighers* against *Vice* and *Folly* in others, who are not without their own *Offences*.—The Palladium Extraordinary, 1763, p. 96.

1290. A note in the accounts of King's School, Ely, dated Lady Day 1609, states that

"Mr. Pamplyn, the Headmaster, conceded three shillings and fourpence of his quarter stipend to Mr. Hynde, the undermaster, because Mr. Pamplyn 'would not teache argybra'."

Apparently Mr. Hynde would and did teach it! We should suppose that algebra was a subject rarely met with in English schools at this date.—*Church Quarterly Review*, Jan. 1939. [Per Rev. E. M. Radford.]

REVIEWS.

Einführung in die Algebraische Geometrie. By B. L. VAN DER WAERDEN. Pp. vii, 247. RM. 18; geb. RM. 19.50. 1939. Die Grundlehren der Mathematischen Wissenschaften, 51. (Springer)

About ten years ago, van der Waerden, already eminent as an algebraist, began, in a series of papers in the *Mathematische Annalen*, to create rigorous foundations for algebraic geometry. The implication—that there was something unsound in the magnificent structure of Italian geometry—was vigorously contested by Severi. Fortunately, van der Waerden continued his researches, but with the implicit sub-title, " An algebraist looks at algebraic geometry ".

With increasing knowledge of the powerful methods of the Italian school, he has gladly modified his own methods. Ideal-theory, the weapon of attack in his first papers, he has found almost completely unnecessary, a change of front which one algebraic geometer, at least, hailed with a sigh of relief! As a result of the experience gained in writing these papers, and in giving various courses of lectures, Professor van der Waerden has produced a work which must sooner or later find a place on every geometer's bookshelves.

The knowledge of algebra required for a complete understanding of this volume is not extensive. The style, as readers of *Moderne Algebra* would expect, is excellent. Also, free use is made of methods and ideas which, at first sight, seem to lie outside the algebraic domain. For example, power series occur frequently, and everywhere-finite differentials are discussed. But, as the preface explains, certain ideas in algebraic geometry are inevitably associated with certain methods, and the objection that these methods are non-algebraic is easily overcome.

One result of this catholicity of outlook is that many of the proofs are very simple. An example is the proof of the invariance of the genus, in which use is made of a property of the everywhere-finite differential. It is evident, in fact, that the author's concern has been to produce a textbook not only rigorous but readable.

There are various worked examples in the text, and exercises follow many of the sections, but to keep the size of the volume within reasonable bounds only the essential elements of algebraic geometry are treated. It must have been with great reluctance that the author decided to omit his delicate theorem on algebraic correspondences, and the ingenious deduction, " The intersection of two algebraic manifolds M_r and M_s of dimensions r and s in the projective space S_n has no component of a dimension $< r+s-n$ ". (Familiar as this theorem may sound, it can easily be verified that the proof in Bertini's *Geometria Proiettiva degli Iperspazi* is insufficient, and the proof given by Severi in *Rendiconti Lincei*, 17, (1933), 3–10, is unsound.)

The following is a very brief outline of the contents. Chapter I contains the elements of the projective geometry of n-dimensional space. Chapter II deals with algebraic functions, their simplest properties and their expansions. Chapter III is entitled " Plane algebraic curves ", and deals amongst other matters of importance with branches, the decomposition of singularities, and the invariance of the genus. Chapters IV and V deal with algebraic manifolds, algebraic correspondences with applications, and the idea of the " attached " form of an algebraic manifold. These two chapters alone would justify the publication of the volume, including as they do van der Waerden's most characteristic work. The concept of multiplicity is fully dealt with in Chapter VI, linear series in Chapter VII. We find Noether's fundamental theorem and its consequences in Chapter VIII, and the analysis of the singu-

larities of plane curves in Chapter IX. An appendix to Chapter IV contains a detailed account of algebraic manifolds as topological constructs.

One slight blemish in the book is the lack of an index. It is also puzzling to find the name of Study attached to a simple lemma, but no mention of Corrado Segre in Chapter I. But there is no accepted rule in such matters.

Finally, this is a book both for the experienced algebraic geometer and for the large number of mathematicians who would like to know something of algebraic geometry, but have found one-volume textbooks like Severi's *Trattato* or *Vorlesungen* unsatisfactory. Van der Waerden's *Einführung* should have a great influence on the future of algebraic geometry.

D. PEDOE.

Die Pellsche Gleichung. By W. WEBER. Pp. vii, 151. RM. 5. 1939. *Deutsche Mathematik*, Beiheft 1. (Hirzel, Leipzig)

Everybody who wants to study this book ought first to read the preface very carefully, where the author explains quite explicitly his aims. The book is not written for beginners, but for experienced mathematicians. At the same time the author aims at a representation of the more elementary facts only.

The result is that (*a*) the existence of a solution of Pell's equation and (*b*) the connection between quadratic forms and quadratic fields is taken for granted, while on the other hand it is never stated that $x^2 - py^2 = -1$ has a solution if $p \equiv 1 \pmod 4$. As the author himself explains: "So wird z. B. die Bedeutung des gegenseitigen Restverhalten der Primteiler von D für die Lösbarkeit der Gleichung $t^2 - Du^2 = -1$ völlig verschwiegen, nicht nur weil sich die Lösbarkeitsfrage auf diesem Wege wohl gar nicht völlig wird klären lassen."

Within these limits the author has produced a carefully written treatise on how to handle Pell's equation (and Pell's equation only). At the end of the book a very useful list of numerical tables is given. H. A. H.

Sur les égalités multigrades. By A. GLODEN. Pp. 91. 25 fr. belges. 1938. (Beffort, Luxembourg)

The subject of this book is also known under the name of Tarry's problem. It deals with identities of the type

$$x_1^k + \ldots + x_s^k = y_1^k + \ldots + y_t^k$$

where k and the x, y are positive integers. The problem is, if s, t, $k_1, \ldots, k_n$ are given, to find $x_1, \ldots, x_s, y_1, \ldots, y_t$ such that the above identity holds for $k = k_1, \ldots, k_n$. To give an example, for $s = t = 4$, $k_1 = 1$, $k_2 = 2$, $k_3 = 3$ we find a solution

$$1^k + 8^k + 10^k + 17^k = 2^k + 5^k + 13^k + 16^k, \text{ for } k = 1, 2 \text{ and } 3.$$

The subject can be approached in two different ways. 1. One may ask whether identities exist for given s, t, $k_1, \ldots, k_n$. 2. One may try to collect as many identities as possible. The author has taken the second course, and as a result has produced a book that is extremely easy to read and which will delight anybody who loves integers for their own sake.

The expert in the theory of numbers should be warned against taking a superior highbrow attitude, because he may find it very hard work to discover, or even to verify, some of the examples.

It is to be regretted that the bibliography at the end of the book is far from complete. H. A. H.

General Mathematics. By C. H. CURRIER, E. E. WATSON and J. S. FRAME. Pp. ix, 382. 13s. 1939. (Macmillan, New York)

On reading most chapters of this book, one obtains the impression—"Excellent—what there is of it". But the range of subjects dealt with precludes over much attention being given to any part, and detail is consequently missing. Calculus (differentiation and elementary integration), Coordinate Geometry (equations of the conic sections and the tangents to them, the line in space and surfaces of revolution), Algebra (graphs, theory of equations, progressions and series, interest formulas, permutations and combinations), and Trigonometry (solution of triangles, the exponential function and complex numbers) all find a place in the first 325 pages, while the book concludes with a chapter on Theory of Measurements, and ten tables, some of a type less commonly found. They include an American Experience Mortality Table, Annuity Table and (without differences) trigonometrical function tables for ten-minute intervals. The latter perhaps could have been omitted in favour of more of the unusual type.

The book is well written and printed; historical notes, which are inserted very freely, add much to the interest of the text and should suggest fresh lines of attack to various parts of the work for those who desire such methods. A good index at the end includes references to persons and places named in these historical notes, as well as to the usual text.

Criticism, in the main, can only be directed to the fact that so much, of necessity, has been omitted, although, of course, this was the authors' intention. The one portion in which the text falls below a good standard is in the introductory Calculus chapter. The authors do not seem at all happy in their work on limits and the whole of the subsequent work on differentiation is apt to be rocky when on this poor foundation.

It is a pity that examples involving the binomial theorem are placed in Chapter VII (*e.g.* page 148), when the theorem itself is not dealt with until Chapter IX. Again, in the chapter on progressions, there are definitions of "arithmetic means" and of "geometric means", yet in the later chapter of Theory of Measurement, definitions of arithmetic mean and geometric mean are given which are likely to confuse the student who has learned the earlier phrases. Omissions strike one particularly in this latter chapter on measurements, which would have profited by more attention to the question of "probable error", and in the chapter on series where it is stated that "2 is said to be the sum of the infinite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$.... and this series is said to be *convergent*", but no conditions are stated for convergency and, indeed, the subject is not again referred to.

The fact that the book is American written rarely intrudes, although such things as 3d for third (page 197) and the word "die" as the singular of dice in examples on page 228 and elsewhere look oddly like misprints, of which, in fact, none is evident. The word "grade" has led the authors into some confusion. Discussing the slope of a line, they state that "grade" is "more often used in common parlance" and is generally expressed as a percentage. Yet in a very early exercise on plotting graphs (*e.g.* page 4) a question (2) reads, "the following grades were made by students and their results are given as a percentage". The word is similarly used when curves are drawn in the "Theory of Measurements" chapter. To draw a graph in which percentage grades are plotted along one axis and then be requested to find the percentage grade of the resulting curve is, to say the least, avoidable.

The questions generally, however, are well chosen and frequently refer to practical applications. Questions on the parabola, for example, include several regarding spans of bridges; those on the ellipse include references to

the design of flower beds, and those on the hyperbola to Boyle's Law and Ohm's Law.

Incidentally, the preface suggests that the exercises were the chief reason for a new edition of the book (for this is a revised version of a text first issued some ten years ago). The authors state that "since lists of solutions to problems are sometimes handed down from one college generation to another, it seemed advisable to publish a revised edition in which the list of problems would be considerably altered and amplified". This cannot have been the only cause of a fresh edition being demanded. The book should be popular amongst those who desire to follow a course similar to that in it and, for such, the volume can be readily recommended. F. W. K.

Analytic Geometry. By ROSCOE WOODS. Pp. xiii, 294. 10s. 1939. (Macmillan, New York)

This volume contains 223 pages devoted to plane geometry, 50 to solid, with a good index. Intended for use "in an introductory course such as is usually given in college and technical schools", it follows in general the normal order of work. After a chapter on fundamental concepts and formulas, there follow sections on the line, the circle, the parabola, ellipse and hyperbola. Then comes a chapter on the transformation of coordinates, and this seems better placed here than occurring earlier, as in many books. The advantage of the transformations, and the identifications of the resulting equations are much clearer.

On the other hand, there seems little gain in setting work on polar coordinates so late as Chapter IX in the book, and, deliberately, it seems, neglecting to emphasis the link between them and rectangular axes. (Incidentally axes other than rectangular are not mentioned in the book.) The author claims as his conviction that "the concept and methods of polar coordinates serve a more useful purpose" when learnt "without reference to any other coordinate system". This unusual point of view is all the more strange since, in the three-dimensional work, rectangular, polar, spherical and cylindrical coordinates are introduced concurrently, and the relations between them pointed out.

A similar disparity between the sections is noted regarding the conditions for parallel and perpendicular lines and planes. These conditions in the three-dimensional section are derived from the expression for angle between planes, but in the earlier work obtained directly.

Another unusual point in the book, and this time a good one, is the frequent use of the interrogation—why? After facts have been collected in a paragraph, a conclusion is stated and then follows the word—why? This encourages a little thought on the part of the student, who is, indeed, sometimes asked to draw his own conclusions. One might say it is a case of presenting the spoon and the food but leaving the patient to deal with them.

Chapters not already mentioned deal with geometric properties of the conic, including descriptions of methods of constructing various figures, and with the second degree equation, which latter deals with the invariants written as I, C, J, D, and mentions briefly and for the first time diameters and conjugate diameters of a conic.

The Solid Geometry section follows briefly a conventional course, chapters being devoted to Fundamental concepts, the Plane and the Line, and Surfaces and Curves. The latter contains illustrations (presumably by photographs) of wire models.

The serious failing of the book appears to be in its examples. The worked questions are often mere substitutions in formulas, regardless of the fact that

examples should illustrate much more than this. Work of this type occurs very frequently:

Example 1. Find the equation of the plane whose intercepts are 2, −3, and 4.

Solution. Substituting in (VII) we have $x/2+y/(-3)+z/4=1$. (Page 243.)

We cannot feel this to be a "solution" of anything.

Something of the same fault is found in the exercises, *e.g.* after a very short discussion on the path of a projectile (in which the essential equation thereof is assumed) some half-a-dozen questions are set to be worked by the student. Each can be done by putting numbers in a formula.

Answers are, incidentally, given to the odd-numbered exercises only. In general, provided succeeding "odd and even" numbered questions are similar, and an attempt appears to have been made to fulfil this condition, this may have some advantages. In any case, volumes containing full answers are available.

On points of detail, one might notice that the figure on page 125 contains $P(x, y)$, $Q(x, y)$ as two distinct points. This is a thing which could be avoided. Again, the phrase "nappe of a cone" is employed with no explanation of what constitutes a nappe. The word appears to be common in American books but far from so in English.

The volume is well printed and set in a slightly larger type than is usual. This gives the book an attractive appearance, but may be the reason why it appears to contain less than one might expect. F. W. K.

Addenda to "Rara Arithmetica". By DAVID EUGENE SMITH. Pp. x, 52. 10s. 1939. (Ginn)

Students who use *Rara Arithmetica*, 1908, know that it owes much of its value as a history of Renaissance arithmetic to the descriptions it contains of the European arithmetical books of the fifteenth and sixteenth centuries in the library of Mr. G. A. Plimpton. Mr. Plimpton was adding to his books and manuscripts when *Rara Arithmetica* was being prepared, and continued to do so until his death in 1936, after which the library was presented to Columbia University. *Addenda to Rara Arithmetica* brings the catalogue of the early arithmetics in the library up to this date.

Each item in *Addenda to Rara Arithmetica* is plainly marked with the number of the page in *Rara Arithmetica* to which it is related. The two books can thus be read together with ease and the arrangement, chronological by first editions, is preserved. Many of the addenda list the works and editions which have been brought to notice since *Rara Arithmetica* was prepared or give additional information about an author, or the date of a first edition.

The longer items describe the newly acquired arithmetics of the Plimpton collection. The particulars of each book are set out in the same concise order as in *Rara Arithmetica*, and there are facsimiles of title-pages and other pages showing illustrations of interest.

An index of names, places and subjects is given. F. A. Y.

A School Certificate Trigonometry. By W. G. BORCHARDT. Pp. vii, 307, xxxiii, xxxix. 4s., with or without answers. In two parts, 2s. 6d. each, with or without answers. 1939. (Rivingtons)

This is a very useful book by a well-known author. The trigonometrical ratios are introduced in pairs and plenty of examples enable the beginner to get a thorough grasp on one pair (for example, sine and cosine) before going on to the next (tangent and cotangent). The treatment of problems in three dimensions is clear and the diagrams will help those students who find difficulty in visualising their work in three dimensions.

Part 1 includes all the necessary formulae for the complete solution of the triangle, including formulae for the radii of circles connected with the triangle. Part 2 includes compound angles, inverse functions, sums and products, equations and elimination.

Throughout the book the work is clearly arranged and there are abundant examples. This book can be recommended for all work in trigonometry from the beginnings to the Higher School Certificate stage. E. E. I.

A First Geometry. By C. G. HAYTER and M. J. G. HEARLEY. Pp. 193. Without answers, 2s. 9d.; with answers, 3s. 1939. (Harrap)

A delightful and interesting first-year course in geometry is set out in this book. The introduction deals briefly with the historical development of geometry and is made particularly attractive to the beginner by the photographs which it contains. The pupil is introduced to the properties of angles, parallels and the simple ruler and compass constructions by means of well-graded examples which are constantly widening his experience and range of geometrical ideas. Field work, measurement of areas and scale drawing are treated very fully and from a practical point of view. Deserving special mention is the section of the book which is devoted to symmetry because of its original and detailed treatment.

A special feature of the book is the introduction at the very beginning of the idea of coordinates under the heading of point plotting and the constant revision of coordinates by means of exercises requiring their use at different stages of the book.

Although no formal geometry is attempted, the content is arranged in such a way that new ideas emphasising the fundamental truths of geometry are being brought before the pupil constantly and at the same time greater practical skill and accuracy in the use of instruments is being acquired. After such a course the pupil should be ready to begin more formal work in geometry. H. B.

A New Geometry for Schools. Stage A and Stage B. By C. V. DURELL. Pp. xvi, 571, xxii. 5s. 6d. Stage A: 1s. 6d. Stage B, Parts I-III: 4s 6d. Stage B, Part I: 1s. 9d. Part II: 2s. 6d.; Part III: 1s. 6d.; Parts I and II: 3s. 9d. 1939. (Bell)

Exercises and Theorems in Geometry. By C. V. DURELL. Pp. xvi, 568, xxii. 5s. 6d.; Stage A: 1s. 6d.; Stage B, I-III: 4s. 6d.; Stage B, I: 1s 9d.; II: 2s. 6d.; III: 1s. 6d.; I-II: 3s. 9d. 1939. (Bell)

Mr Durell's new books, covering the whole of the School Certificate Course in Geometry, once again put the author well into the lead in the field of geometrical teaching. The two books are alike in content but are differently arranged. In the case of the *New Geometry for Schools*, all the formal proofs of Theorems and Constructions are presented at the appropriate places in the text, whereas in *Exercises and Theorems*, the Exercises, Constructions and Theorems appear grouped together in separate sections of the book, in the same manner as was adopted in the author's earlier textbook, *Concise Geometry*.

The books are a great advance in geometrical teaching. They embody many of the suggestions put forward in the Second Report on the Teaching of Geometry published by the Mathematical Association and are extraordinarily well planned and graded. Stage A introduces the pupil to ruler and compass constructions, scale drawing and the essential properties of angles, parallels, congruence and elementary similarity. The introduction of trigonometrical ratios into Stage A is unusual and very interesting in view of the later

treatment. Stage B covers the more formal Geometry to School Certificate. The books are both admirably set out. The exercises, which are numerous, are very carefully selected and graded and offer ample scope for all types of pupils. The examples in each group of exercises are classified into three sections. The first type indicates the minimum to be done by all pupils; the second type is for extra practice in further examples of the same standard, and the third type consists of examples essentially constructed for the brighter pupils. All three types are extensively illustrated by diagrams and should be very helpful to both teacher and taught. Examples for oral discussion before the exercises are attempted are a welcome feature of the book.

The introduction of trigonometry and its use in giving an alternative treatment of geometrical properties by expressing them in trigonometrical form wherever possible are striking features of the book. This treatment will be thoroughly welcomed and appreciated by those teachers who feel that there should be some fusion of the teaching of trigonometrical ratios with that of geometry.

Other points which deserve special mention are the introduction of examples involving the application of the properties of plane geometry to solid figures, the use of small letters for angles, and a more detailed treatment of loci. Both books should prove excellent textbooks for all School Certificate candidates.

H. B.

Wahrscheinlichkeitsrechnung für Nichtmathematiker. By K. Dörge. Pp. 113. RM. 6. 1939. (Walter de Gruyter, Berlin)

The name of Professor Dörge is associated with an attempt to improve the logical basis of von Mises's theory of probability. As was explained in some detail in the review of von Mises's book (*Gazette* XXIII, July 1939, pp. 309-310), probability was defined with reference to a Collective (Kollektiv), which is an unlimited series of observations such that the relative frequency of a particular attribute tends to a fixed limit unaffected by any place selection. Doubts have been expressed whether there really exists any series of observations satisfying these somewhat stringent conditions. Dörge gets over this difficulty by giving categorical rules for the construction of such collectives. He starts with a certain finite number of sequences and also a certain finite number of place selections, both explicitly defined by formulae. Certain operations called selection, partition, and sampling are then performed repeatedly, until no new elements arise. The result forms what may be called a closed corps of collectives and place selections, and the theory of probability is then applied to this corps. As an abstract theory it leaves nothing to be desired, but the difficulty arises when we try to find phenomena of the external world which have the same properties as our carefully constructed abstract system. An account of this work will be found in the *Jahresbericht der deutschen Mathematiker-Vereinigung* (43, 1934; 39-47), and, more fully, in *Mathematische Zeitschrift* (32, 1930; 232-258 and 40, 1935; 161-193).

The book under review has little to do with this abstract theory; in fact the index does not even contain the name von Mises or the term Kollektiv. After an introduction dealing with the sex distribution of entries in a register of births, and a chapter on sequences and limits, probability is defined as the limit of a relative frequency, *provided that this limit exists*. The hypothesis is then made that the observed values of a quantity measured in a scientific research form a sequence with all the properties required. A second hypothesis is that, in cases of physical symmetry, we have equal probabilities. This, of course, is the old "equally likely" theory as interpreted by the Principle of Sufficient Reason. After these foundations have been laid down

the rest of the work proceeds more or less on the usual lines. The last of the five chapters deals with the mean, the standard deviation, Tschebyscheff's inequality, and the law of large numbers. An appendix gives the rigorous proof of certain theorems on limits.

The title is somewhat misleading, as the book is certainly not suitable for non-mathematicians, in the ordinary sense of the word. The reader is supposed, although ignorant of mathematical results, to be capable of grasping them as soon as they are presented to him, to be undismayed by a crowd of mathematical symbols bristling with suffixes and affixes, and to follow the rather subtle reasoning involved in the discussion of the theory of limits. As a matter of fact, as the preface reveals, the book is really intended for statisticians, and has developed out of a course of lectures addressed to them at the University of Cologne. It is likely to be useful chiefly to the special class for whom it was originally designed. H. T. H. P.

The Geometry of Determinantal Loci. By T. G. Room. Pp. xxviii, 483. 42s. 1938. (Cambridge)

This long-expected book from Professor Room is, on the technical side at least, one of the most important recent additions to the literature of Projective Geometry. It had been hoped that Mr. Room would produce a short book dealing with the technique which he has been developing in many papers, but he has elected to write a comprehensive volume in which, not content with merely establishing his methods of attack, he has illustrated those methods by applications over an extensive field. The fate of a large book is always problematical; all geometers will welcome the mine of information which has been opened up in a systematic way, but the influence of an elaborate treatise on current mathematical practice is often less than it ought to be. Doubtfully intending readers should know that large portions of the book, which are specified by the author, are of thoroughly general interest, though some chapters are clearly for the worker who is interested in particular questions. For example, a chapter is devoted to generalisations of configurations related to the Segre cubic variety; this, though containing some of the most striking work in the volume, is scarcely relevant to the main purpose of the book, being indeed more of the nature of a mature contribution to a periodical. On the other hand the methods, here fully developed, of investigating a manifold whose equations are given in determinantal form are a necessary part of the equipment of the most modest student of higher space. "Practically all the loci about the projective properties of which anything is known either are included in the class of Determinantal Loci, or are closely connected with it."

The arrangement of the book has evidently received most careful thought. It is divided into three parts, the first of which is an exposition of notation and procedure, the second and longest is an application in detail to general matrices of certain types (mainly Veronesians, Grassmannians and rational scrolls), and the third is a discussion of determinantal primals of the fourth order in four dimensions with an examination of many particular cases. The advice given to the reader to begin almost at the end of the book speaks for itself. The teacher and the reader have an advantage over the writer; they begin naturally with a particular case, settle down to think in particular dimensions and with a particular configuration, untrammelled by the exceptions and inequality conditions of the general case, and in so far as these exceptions and conditions are trivial they master the whole principle and may well find the subsequent generalising easy or even tedious. But the writer of a book must accept the logic of his subject and can only proceed, economi-

cally, from general to particular. When, as in this case, he shows that he appreciates the obvious dilemma and writes the last part as a possible first part, he places his readers under an unwonted obligation. Also he has written, apparently as an afterthought, a short introduction to indicate the tenor of the book in terms of familiar cases such as quadric and cubic surfaces, although this sketch is rather too brief to open up the argument. Moreover every piece of general reasoning is followed by particular cases as examples, and these have often an exciting tail to them which prevents them from being in the nature of an anticlimax.

The language of the enunciations of the theorems is geometrical and sometimes intricate, but the argument usually depends on simple algebra. A determinantal locus is a manifold which arises from the statement that a matrix x is of rank r, the elements of x being regarded as linear functions of the coordinates of a point. (The word "matrix" is used in the sense of scheme or array and not in the sense of a matrix algebra. The elements are sufficiently general functions of the coordinates and no question of the character or invariant factors comes in.) This matrix can be expressed as the product of two matrices α with r columns and β with r rows, and $x=\alpha\beta$ serves as a kind of parametric form for the manifold. In the simplest case where the elements of x are linearly independent there are immediately systems of α-spaces and β-spaces on the manifold which behave somewhat as the systems of generators on a quadric. The construction of either of these systems is termed a projective generation of the manifold. When the elements of x have linear relations between them, the manifold is a section of this *key*-manifold. The subject of the book is the study of manifolds and configurations $x=\alpha\beta$, and especially the relations between all those which arise from the same p by q matrix, including among them the cases $p\leqslant r<q$ where there is no proper manifold and no determinantal equation at all (*e.g.* the chords of a twisted cubic from a 2 by 3 matrix). As a sample of such relations may be quoted that the linear spaces, other than α and β spaces, on the key-manifold of rank r are appropriate joins of α-spaces and β-spaces belonging to key-manifolds of rank lower than r. In section these sub-manifolds may or may not appear, but the traces of the joins yield linear spaces on the section of the original manifold.

Few references are given to older work on the subject save by way of the Encyclopedia article; the author rather suggests that he is merely repeating with a certain nomenclature work that is already classical. This, however, is far from being the case. The notation indeed is all-important, for on its abstraction depends a great deal of the power of generalisation. That the symbolisation is adequate is demonstrated by the fact that it is carried unchanged throughout the book in spite of the very different calls that are made upon it. It is to be hoped that it will percolate through to more elementary treatises. But apart from notation there are many other selections and variations required for effective generalisation. One example must suffice for illustration. The general cubic surface is given by the vanishing of the determinant of a 3 by 3 matrix x. If λ, μ denote single row and single column matrices, $\lambda x=0$ is the equation of three projective stars and the common point P of their corresponding planes generates the surface. Similarly $x\mu=0$ gives a point Q generating the surface, P and Q being in this case the α and β spaces referred to above. For a particular value a of λ the intersection P may be a line instead of a point, in which case there is a matrix b such that $axb\equiv 0$; hence there is a value b of μ presenting Q as a line in the complementary generation. As is easily seen the identity occurs on six occasions; there are six paired lines $a_i x=0$, $xb_i=0$ on the surface. The line $a_1x=0$ meets the line $xb_2=0$ since there is the further obvious relation $(a_1x)b_2=a_1(xb_2)$

between the equations, but no such identical relation holds between $a_1x=0$ and $xb_1=0$. Hence the a's and b's form a double-six. This simple argument has the merit of emphasising the predominance of the double-six on the surface; as a result it goes back to Grassmann. But the beauty of this approach lies in the capability of immediate generalisation to a whole series of pairing theorems on projectively generated manifolds, and to the establishing of the existence of double-tens of lines and planes in four dimensions, double-twenties of planes in five, and so on. This was first shown by Mr. Room in one of his earliest papers, and as the argument runs like a thread all through the present book it may well be regarded as the germ from which the book has sprung. The most straightforward generalisation is to the case of a primal given by the vanishing of a determinant of a q by q matrix in q dimensions. Here the a-lines form a family, and paired with each line is a b-line of another family. Just as the c-line on the cubic surface has an equation of the form $lxm=0$ where l is a 2-row and m a 2-column matrix, so on the primal there are $q-2$ further families of lines, each line given by an equation $lxm=0$ where l has $q-i$ rows and m has $i+1$ columns. These are in fact the traces of the joins of α and β spaces on the key-manifolds to which reference has been made above.

This is only an indication of one of the many lines of thought which serve to systematise a mass of material, but it may suffice to show that a close study of the main chapters will amply repay anyone interested in generalised configurations. Mr. Room's references to the work of his contemporaries are full and generous; indeed the only apparent signs of overloading occur as the result of his desire to give full weight to any work of his colleagues which has a bearing on his own. The excellence of the work thus becomes a testimony to the fertility of the Cambridge school of geometry during the last fifteen years and to the inspiration, much alluded to in its pages, of its director. P. F.

Physical and Dynamical Meteorolgy. By D. BRUNT. Second edition. Pp. xxiv, 428. 25s. 1939. (Cambridge)

The appearance of a second edition of *Physical and Dynamical Meteorology* is welcome as it has afforded the opportunity of adding the more recent developments in the theory of Meteorology. This has entailed chiefly the addition of a paragraph here, the elucidation of a point there, but taking the book as a whole there are few fundamental changes from the former edition, though the chapter on Turbulence in the Atmosphere has been largely rearranged and rewritten.

The exposition of Meteorology in such a book as this invites the reader to examine the shortcomings of the present statement of the Science of Meteorology. To say that the first fourteen chapters of the book which deal with radiation, turbulence, the thermodynamics of the atmosphere and the exposition of the equations of motion are far stronger and more satisfactory than the last five probably reaches to fundamentals, for the later ones are describing the bases of forecasting weather.

The author in his preface states that the book does not profess to be a textbook of weather forecasting; how many would welcome such a textbook, but one is led to feel that the time for such a book is not yet. Much work has to be done. For instance, the exact account of change of pressure has not yet been formulated on dynamical principles in any quantitative manner, except to some extent in the discussion of the monsoons due to Jeffreys. For the moving pressure systems of both tropical and extra-tropical latitudes there is no such statement. When it is remembered that the day-to-day forecasts are based primarily on the visualisation of the future pressure distribution and

the translation of that picture into weather, it will be realised under what a great handicap the forecaster labours and indeed why the author alludes to the Art (rather than the science) of Forecasting.

Stated baldly, in its present development the forecasting of pressure changes is based on a number of empirical rules, derived in many cases from sound physical principles, but nevertheless empirical in that no estimate of a quantitative value can be given that is not entirely dependent on the experience of the previous few hours. There are, it is true, some mathematical expressions of those empirical rules, notably Pettersen's well-known attempt to determine the future pressure from the behaviour of pressure in the past, and it is rather surprising that in this book there should be no account of this work. But such attempts have to be supplemented by the forecaster's skill in weighing up the dynamical changes which are going on.

Admittedly the dynamical problem involves a more extensive and exact knowledge of the day-to-day variations of temperature in the upper air than is at the moment available, but with the international efforts which were taking place before the outbreak of war the data, it might be hoped, would not remain so tantalisingly insufficient. Nevertheless if these new data are to be used it is essential that forethought shall have provided the mechanism for their use. The fundamentals are written plain in the equations of motion and the equation of continuity. The opportunity lies open for some research worker to develop a method.

Perhaps we may look forward to a third edition of *Physical and Dynamical Meteorology* in which the last five chapters, somewhat enlarged, will outline sound methods of computing future weather maps based on physical and dynamical principles. Such an addition would round off the whole, and bring the student's knowledge, gained in the first half of the book, into continuity with the latter part.

One further point; one seeks in vain for the statement of the physical laws governing visibility. Yet to many students that is a meteorological factor of prime importance. There may be no obvious niche into which such a discussion would fit, but even at the expense of a digression it should have its place and thus perhaps allow for a somewhat greater emphasis on the formation of fog.

C. S. DURST.

The Principles of Statistical Mechanics. By R. C. TOLMAN. Pp. xix, 661. 40s. 1938. International series of monographs on physics. (Oxford)

One may perhaps be allowed to preface a notice of this book by deploring, with all respect to its distinguished author, a feature which it shares with a number of recent books on various branches of mathematical physics. This is the amount of space given to an outline of another subject, of which the results are required for the purposes in hand, even though standard modern texts on that subject are readily available. In the present case a chapter of 27 pages is given to the elements of classical mechanics, and two whole chapters and the greater part of a third covering altogether about 190 pages are devoted to pure quantum mechanics. Now it did so happen that when statistical mechanics was in process of attaining its modern form, quantum mechanics was only just being developed. So a writer on the former theory had then good reason for digressing in order to derive the quantum mechanical results which he needed, since there was but small chance that his readers would be familiar with them. But there are now in existence standard accounts of quantum mechanics, written from various standpoints, by almost all the leading authorities on the subject. Moreover, anyone wishing to make a proper study of statistical mechanics now knows that he is expected to have an adequate

knowledge of quantum theory, and he should be in a position to make intelligent reference to any of these standard works. Consequently, although the whole basis of statistical mechanics is now inseparably bound up with that of quantum mechanics, it ought to be possible for the author of a book like the present one to adapt his treatment, with no substantial modification, in such a way that suitable allusions to one or more well-known texts on general quantum mechanics would suffice in place of giving a fresh development of the whole subject.

In just the same way it is not necessary that every new book on mechanics or electromagnetic theory should contain an account of vector analysis, or that every new book on general relativity should contain one on tensor calculus, or that every new book on astrophysics should comprise also treatises on thermodynamics, radiation theory, and quantum theory. The general point involved is by no means trivial. For we are constantly reminded that the present time is one of cooperation in science. Surely an elementary sort of cooperation would be for authors of new books to take full advantage of existing works by other writers in formulating the presentation of their own material.

Indeed one might go further and urge that more advantage should also be taken of the accessibility of standard journals containing the original detailed working out of individual results. For presumably the main object of a book of principles like Professor Tolman's is to present the logical sequence of the fundamental ideas of the subject. Such a book is designed not so much to derive results as to fit them into an orderly pattern. The more therefore the author is willing to refer the reader elsewhere for non-fundamental details the more boldly will the main lines of the pattern stand out. Incidentally, the reader will also be given the healthy exercise of going to original sources for parts of the work.

As has already been suggested, these criticisms might equally well be levelled against other books besides Professor Tolman's. But they are more particularly suggested in this case because its 650 pages represent a longish journey, and one naturally enquires if some short cuts would not have been permissible. Even if they were not, one would feel inclined to suggest that, had Professor Tolman seen fit to introduce more of the spirit of adventure and discovery, he would have persuaded us to follow him all the more willingly. To tell us each time, as he does, just what lies round each turning of the road before we get there may indeed make for safety, but it spoils the excitement. Also Professor Tolman supplies excellent reasons for taking the road he selects; but we should have liked to explore some of the others. For instance, he might have permitted us to see more fully, in company with Birkhoff, Hopf, von Neumann, and others, how far the ergodic hypothesis would carry us.

Professor Tolman's book may be regarded, broadly speaking, as a modern version of the classical work of Willard Gibbs. The two peaks in the statistical theory are the generalised Boltzmann H-theorems for classical and quantum mechanical systems. Their derivations spring from the hypothesis of equal *a priori* probabilities for equal regions in the phase-space, and its quantum theory analogue. In the treatment of quantum mechanical systems full account is, of course, taken of the limitations imposed by the Uncertainty Principle upon the type of information we may assume to be available, as well as the limitations characteristic of the essentially statistical descriptions. The culmination of the work is the statistical explanation of the principles of thermodynamics. With this in view, the author develops the statistical theory so as to take fuller account than is usually done of the " surroundings " of the

system of interest, so that its circumstances can finally be the more accurately matched with those appropriate to actual thermodynamical observations. In such work Gibbs's concepts of the canonical ensemble and the grand canonical ensemble play an essential part. There are also important sections of the book dealing with fluctuations of a system about its mean state. The treatment throughout may be characterised as being extremely thorough, on the level of rigour chosen by the author, rather than very profound.

Except in the last chapter, where they appear more as illustrations of the scope of the results that may be achieved than for the sake of the results themselves, applications to particular physical cases are avoided. For such applications Fowler's *Statistical Mechanics* must long remain the standard work. But as a companion work on the general principles of the subject, Professor Tolman's book will undoubtedly take an important place, in spite of the sort of criticism to which it seems to the reviewer to be exposed. W. H. McCrea.

Physique stellaire. Essai de synthèse. By E. Sevin. Pp. 80. 25 fr. 1939. (Gauthier-Villars)

This tract contains a sketch of the current theory of stellar structure, followed by certain criticisms of the theory together with M. Sevin's proposed alternative treatment, and finally an outline of his theory of stellar energy-generation and stellar evolution. His criticisms are based on misconceptions which arise when he attempts to view the same phenomenon both from the microscopical and macroscopical standpoints. Firstly, he considers that current theory contradicts the law of conservation of momentum in predicting an *outward* force due to radiation pressure on the material at any point in a star, depending on the absorption coefficient of the material. For, he argues, since the radiation from a star carries away momentum radially outwards, the recoil momentum ought to produce an *inward* force on the stellar material, depending only on the net emission of radiation and not upon the absorption coefficient. But this criticism is fallacious, since the net emission of radiation from the material in any small volume element anywhere in the star is always taken to be symmetrical in all directions and so the emission itself produces no net force upon the material. However, owing to the spherical symmetry of the star as a whole, the distribution of energy-sources must produce in the steady state a net outward flux of radiation at any point (other than the centre) and this flux in, so to say, forcing itself out against the opacity of the material tends to distend the star, that is, it produces an outward force depending on the opacity. Secondly, M. Sevin claims to derive a relation additional to those normally used in the theory of stellar equilibrium. This he does by considering in detail the motion of an atom in the stellar gas and the effect of the acceleration due to gravity upon its behaviour in the intervals between collisions with other atoms, and then deducing the statistical consequences. But this should lead merely to a recovery of the equation of hydrostatic equilibrium as employed in current theory, which represents the macroscopic description of the same physical process. The reason why M. Sevin derives a different result is presumably to be found in the approximate character of his method of averaging in order to get statistical results.

The rest of M. Sevin's work is bound up with a particular and, as far as one can see, quite arbitrary hypothesis concerning the production of stellar energy by the annihilation of matter. Actually the time for such arbitrary hypotheses would seem to be past, for it is now becoming possible to employ laboratory data on nuclear processes in order to begin to build up a proper physical theory of energy-production in stars. Work such as that described in an important memoir by H. A. Bethe in the *Physical Review* this year promises considerable

success in this direction. Such being the case, there seems to be at the present stage in the development of astrophysics but little value in speculations not well founded on known physical processes.

The value of M. Sevin's work, one ventures to suggest, is that, in attempting to answer the criticisms of existing theory which it contains, the reader is induced to clarify his ideas about the physical significance of the various steps taken in the theory. W. H. McC.

Les nouvelles méthodes du calcul des probabilités. By L. BACHELIER. Pp. viii, 70. 25 francs. 1939. (Gauthier-Villars)

The title of this book is rather misleading, as the only new methods discussed are those due to the author himself. Most treatises on the subject contain an investigation of the probability of r successes in n trials, and obtain a result involving factorials. They then deduce an approximate result, holding when n is large, by the use of Stirling's asymptotic formula, and finally obtain the Normal law, holding for a continuous distribution. Professor Bachelier's fundamental idea is to reverse this procedure. Instead of starting from a discontinuous distribution, and passing by means of approximation to a continuous one, he starts with a continuous one. This has the advantage that his formulae are exact, instead of being merely approximate. The starting point, that of a game of chance in which the gains and losses are continuous, offers difficulties which are not discussed. With the object of giving as many results as possible, all the proofs have been omitted, and the reader is referred to the author's more extended works. On p. 14 we find a result which looks like a generalisation of the Normal law, and is stated to be the simplest of the author's "hyperasymptotic" formulae, and to be a solution of the fundamental problem. "La formule hyperasymptotique est analogue à celle de Laplace, mais elle en diffère par une propriété essentielle. Elle est exacte." It is tantalising to have such a claim made without being offered the evidence for it. It is also not clear how one can pass from continuous probabilities to the usual discontinuous problems. In the preface, the author puts forward a strong claim for the importance of his work. "... les résultats réellement essentielles. Ceux-ci sont d'ailleurs très nombreux et dépassent de loin ceux que l'on considère comme classiques. Ce Livre, uniquement descriptif, n'étudie que des théories générales, sous une forme réduite il résume une science très étendue ; il n'a d'analogue en aucun pays." H. T. H. P.

The Theory and Practice of Electron Diffraction. By G. P. THOMSON and W. COCHRANE. Pp. xi, 334. 18s. 1939. (Macmillan)

This book deals with experimental technique, mathematical theory, and applications to various problems of surface- and film-structure. It would appear to serve both the general student who wishes to learn something of the present achievements and future possibilities of the method, and also the investigator who requires guidance in some of the more obscure details of interpretation.

A book of this kind is necessarily rather uneven in style, but in compensation the reader gains in many places the special interest of an authoritative survey of the various hypotheses advanced by different workers to account for effects not yet completely understood. Considerable space is devoted to an outline review of the mathematics involved in problems of interpretation, since it is in this rather than in the experimental technique that there are outstanding difficulties. There appears to be some confusion in references to Plates (pp. 37, 76), and Figure 61 (p. 198) is misleading, but the general lay-out of the text is admirably clear. W. H. T.

An Introduction to Modern Statistical Methods. By PAUL R. RIDER. Pp. ix, 220. 13s. 6d. 1939. (John Wiley and Sons, New York; Chapman and Hall)

Methods of Statistical Analysis. By C. H. GOULDEN. Pp. vii, 277. 17s. 6d. 1939. (John Wiley and Sons, New York; Chapman and Hall)

The advances that have been made in the present century by those mathematicians who have studied the theory of statistics have rather tended to put out of date any textbook in which the subject-matter is treated along traditional lines. Books which incorporate the results of recent research are, therefore, to be welcomed. Prof. Rider is a mathematician who has been able to cut short his exposition by use of mathematical terminology, and by an appeal to mathematical ideas, while at the same time he has kept clearly in mind the practical needs of the statistical worker. The work, however, cannot be called mathematical in the true sense of the term, since in most cases, particularly with sampling distributions of statistics and significance tests, results only are stated, and the author is chiefly concerned with showing how the various formulae are applied. Frequency distributions are described without any reference to the theory of probability. This is followed by a treatment of the various averages, and of the standard and mean deviations and moments in general. Following a quite unorthodox order the author then deals with regression, simple, multiple and curvilinear, and with correlation, simple, multiple and partial, before introducing the binomial and normal distributions. He shows how a normal distribution is fitted to observed data, and includes in the same chapter the various tests of significance, including that applicable to the correlation coefficient, and with an interesting discussion of fiducial, or confidence, limits. A further chapter, entitled "Student's Distribution", groups all the tests that are applicable to finite samples of observational data. The "Chi-Square" distribution is then dealt with in its many applications, and the way is then clear for what the author considers, quite rightly, to be an important aspect of the whole subject, the analysis of variance, not only to provide tests for significance in regression and correlation problems, but also as a means of dealing with heterogeneous data grouped into classes. At this point the well-known application to field experimental data in agriculture is met with. This necessitates a special chapter on experimental design, which not only deals with the simple cases, but also expounds lucidly those principles connected with orthogonality which have led to the discovery of many interesting lay-outs, based upon factorial designs, and which also, by the controlled abandonment of orthogonality in certain cases, have brought about a material reduction in the experimental error which enters into the comparisons made. Unfortunately, the author has not gone as far as some would like him to have gone in this direction, but references are given for further reading.

The necessary tables are provided at the end, and each chapter concludes with a number of exercises for the student. If there is one thing that may be criticised, it is the arithmetic, for a number of slips have been detected, *e.g.* at the foot of p. 80, p. 157 (degrees of freedom), pp. 164, 168 (typographical error in Fig. 13), 182 (Table 60) and 187. The book as a whole forms a stimulating introduction to the subject for the moderately equipped mathematician.

Methods of Statistical Analysis is written by an agricultural scientist who is an expert on the subject. He is concerned to provide readers who may already have had some elementary training with a practical knowledge of methods of analysis and some facility in the devices of calculation. The subject-matter is the same as Rider's *Introduction*, and requires the same general understanding of mathematical terminology, but explanations are much fuller, and very much more material is included. The author prefers to deal with tests of

significance for small samples, and with an introductory treatment of the design of experiments, before going on to regression and correlation, but otherwise the order of treatment is the same as in Rider. The book will be particularly indispensable to the agricultural worker, for a long chapter deals with the many ramifications of the Field Plot Test, and makes available in textbook form all the methods that have only hitherto been published in technical papers. The treatment is clear and detailed throughout, and the book is to be heartily commended to those who wish to see these methods fully explained and exemplified. The analysis of covariance has rather better and fuller treatment given to it than has been the case hitherto. Full chapter references are given, and the student is not only provided with many fully-worked examples, but also with exercises. There are tables at the end to be used for providing answers to the various tests of significance.

There is little to find fault with in the theoretical treatment. Points which have emerged on a first reading, and which are perhaps worthy of mention, are the following. On p. 29 the use of the standard errors of g_1 and g_2 is not qualified by the statement that they are not particularly good measures of deviation from hypothetical values, and that possibly a better test would emerge from using Pearson's tables. On p. 56 the value of t given for a difference of regression coefficients is not generally correct when the samples are of unequal sizes. On p. 251 the point should be made that an essential step in the process of covariance is to test the significance of the error regression. Minor points are the wrong spelling of Isserlis on p. 224, and the absence of a reference to his work, and the writing of " Yield Trials " instead of " Field Experimentation " on p. 259. J. W.

Studies in Arithmetic. Reports on investigations relating to present practice and teaching methods in the Primary School. Pp. xx, 144. 5s. 1939. Publications of the Scottish Council for Research in Education XIII. (University of London Press)

Cooperation between teachers and research workers has long been desired and signs are not wanting that it is beginning to be achieved. Teachers are becoming increasingly willing to submit their methods to the test of experiment and scientific investigators are refining the means by which they measure the products of the schools. This volume makes a modest beginning in summarising for teachers of arithmetic some present-day findings of research and it offers to research workers a useful analysis of present-day practices in that subject.

More than three thousand teachers cooperated to make the report possible. The widespread interest which such numbers indicate is more than a justification for the labour and expense incurred. The value of the resulting evidence lies not only in its revelation of problems awaiting solution but also in the increase of professional competence which probably comes to every teacher who cooperates in such an enquiry.

The book is arranged in five sections :

(1) A summary of the time allocated to the teaching of tables, mental arithmetic, mechanical operations and written problems in 3084 schools in Scotland.

(2) An analysis of replies to 400 copies of a questionnaire as to methods used in the teaching of certain representative processes and topics.

(3) A report of an experimental study of the relative difficulty of the basic number facts for pupils of eight and nine years of age.

(4) The record of an experiment in adding upward as compared with adding downward.

(5) A factorial analysis of results of tests in mechanical operations and reasoning arithmetic.

Teachers are apt to regard as unassailable the methods and the time-table to which they have become accustomed. The wide range of variation in such matters which is here shown is therefore a useful preliminary to the scientific study of the problems involved. Judicious mention of the more pressing of these problems adds interest to the record, and selected references to recent studies of time-allocation, long division, fractions, addition, number combinations, etc., increase its usefulness for the student.

The report on the relative difficulty of the basic number facts is probably the most comprehensive that has yet appeared. It should interest teachers in the advantages of judicious postponement of drill, and it should encourage their growing suspicions of the efficacy of traditional routine ways of learning the tables. The experiment on methods of adding (while not apparently conducted with groups paired on the basis of an intelligence test) provides a useful record of some of the precautions which have to be taken in clinical researches of this type.

The book as a whole is a worthy contribution to the preliminary studies on whose foundation a scientific scheme for the teaching of Arithmetic may one day be constructed. Its value would have been even greater had its final chapter contained references to the more recent work of Spearman, Kelley, Thurstone and others on the analysis of arithmetical ability. C. M. F.

Intermediate Solid Geometry. By L. Lines. Chapters I-VIII of *Solid Geometry*. Pp. xv, 131. 2s. 6d. 1939. (Macmillan)

Mr. Lines is the author of an enterprising textbook wherein are included such modern topics of interest as lattices, sphere packs and crystal patterns. The present little volume is a reprint of the first eight chapters of the larger work and deals with the Solid Geometry of the Higher School Certificate course. Though much of the content is necessarily traditional the book presents a refreshing number of novelties and one finds, for instance, theorems on the orthocentric tetrahedron, generators of a regulus and the extension of Menelaus to a skew quadrilateral. Glimpses of coordinate geometry are given from time to time, direction cosines are defined, the equation of a plane derived by projection upon the normal, and several of the exercises are of a nature to link up with analytical work.

The arrangement of the subject-matter is coherent and helpful. The various propositions are grouped into separate chapters under their appropriate headings, parallels, normals and the like, and the numbering conforms to the chapters. Thus 5T6 means Chap. 5, Theorem 6. It is easy, on this system, to memorise the sequence. A distinction is made between "Theorems" and "Worked Examples" and the latter are numbered on their own, again by chapters. Amongst them are many problems of construction, excellent for driving home the theoretical properties; similar problems are set as exercises for the student.

It is a pity, we think, that there is no chapter on the regular polyhedra. Under modern conditions these are dealt with at the very beginning of the Geometry course; but even so, their reappearance and treatment from the advanced point of view is bound to be interesting. Perhaps the author will consider the point for further editions.

If any criticism may be put forward it is that the book, as produced, presents a somewhat congested appearance. Propositions occasionally commence at the bottom of a right-hand page, preventing a bird's-eye view of the reasoning. Some of the figures confuse rather than assist; but here the fault lies in the

technique of suggesting three dimensions. If these defects can be amended without unduly increasing the price, both author and publisher will have to their credit an attractive and valuable little work. H. L.

Mathematical Recreations and Essays. By W. W. ROUSE BALL. Eleventh edition, revised by H. S. M. COXETER. Pp. xvi, 418. 10s. 6d. 1939. (Macmillan)

This is still Rouse Ball, but with a difference. A large part of the tenth edition is reprinted without change; this needs no description here. Three chapters have, however, been omitted entirely, the one on string figures, which Rouse Ball himself had very much cut down after the appearance of his separate book on the subject in 1920, the short chapter on bees and their cells, and the chapter on mechanical recreations. This last, which begins with Zeno's paradoxes and includes a mention of cut and spin, might well be expanded, as Rouse Ball himself suggested, into an independent work; it is about time that someone dealt authoritatively with the mystery of the "new ball" in cricket.

Recent work in the theory of numbers has made a good deal of change necessary in Chapter II; here Dr. Coxeter has had the help of Professor D. H. Lehmer, whose work on the computational side is well known. Similarly, the chapter on map-colouring problems has undergone considerable revision in the light of modern developments in topology.

It is, however, to the entirely new chapter on polyhedra that one turns with the greatest expectations, which are not disappointed. Dr. Coxeter is an unrivalled authority on the subject, and in these 32 pages he gives a masterly description of the five Platonic solids, the thirteen Archimedean solids, the Kepler-Poinsot polyhedra and so on. The diagrams are excellent, and there are two admirable plates from photographs by P. S. Donchian.

Finally, the chapter on cryptographs and ciphers has been replaced by one entitled "Cryptography and Cryptanalysis", written by Abraham Sinkov, of the United States War Department. Rouse Ball admitted that his knowledge of the subject was largely the result of casual reading, and most of his chapter was devoted to historical examples. We now have a thoroughly up-to-date treatment by a professional, which should be of immense service to writers of detective stories. This chapter alone is provided with a list of recent works for further reading.

Rouse Ball's *Recreations* has been deservedly popular since its first appearance in 1892. It may be predicted with confidence that Dr. Coxeter's admirable edition will give it a new lease of life. F. P. W.

Mathematical Snapshots. By H. STEINHAUS. Pp. 135. 10s. 6d. 1938. (Stechert)

We must be grateful to Professor Steinhaus for deserting the abstract subtleties of real variable theory for a while to provide us with this delightful Mathematical Zoo. The metaphor is the author's own; in his preface he fears that his book may prove to be "too scientific for a child and too childish for a mathematician" but maintains that "mathematical objects are sometimes as peculiar as the most exotic beast or bird, and that the time spent in examining them may be well employed". I am quite sure that every teacher who reads this book will agree with the last sentence, at least as far as the mathematical objects which Professor Steinhaus exhibits are concerned.

It is somewhat difficult to describe briefly the contents of the volume. A very large number of illustrations with short commentaries is hardly an adequate indication of the wealth of material and beauty of treatment. A few

samples may be mentioned: knots, a map on a tore needing seven colours, Möbius bands, loxodromes, soap-bubbles, polyhedra, crystals, the conic silhouettes, Crivelli's *Annunciation* and its perspective, tessellations, chessboards, noughts and crosses; these have been picked at random openings. But above all it is the beauty and charm of the illustrations, 180 in all, which make the book one to possess; black and white diagrams, coloured diagrams, anaglyphs, and superb reproductions of photographs. One item which should fascinate most children and most mathematicians is the model of a dodecahedron. The net is in two parts, joined by elastic; it lies flat in the cover-pocket but when taken out springs automatically into solid form; the faces are coloured to exhibit one of the two essentially different four-colour arrangements possible for the dodecahedron.

Teachers who have seen this book are enthusiastic about both its beauty and its stimulating effect. Author, printer, publisher, and the many helpers who worked at the production of the illustrations are to be most warmly congratulated on a splendid piece of work, for which ten and sixpence is an absurdly cheap price. School libraries should order at once. T. A. A. B.

Mathematics for Actuarial Students. I. Elementary differential and integral calculus. Pp. viii, 183. 9s. **II. Finite differences, probability and elementary statistics.** Pp. xiii, 339. 25s. 1939. (Institute of Actuaries; Cambridge University Press)

This work has been designed to take the place of the previous publication issued in 1931 under the title *Actuarial Mathematics*. It is a great improvement on the previous work, the improvement beginning with the title. The first part contains a short chapter on elementary trigonometry; the remainder of the volume is devoted to that part of the calculus which is considered to be essential for students before they begin their professional studies. This part calls for little notice, except to say that the part of trigonometry most useful in the study of the calculus appears to be excluded.

In the second part, finite differences take up the first 204 pages. This is by far the most distinctive portion of the work and makes the book worthy of consideration by all persons interested in questions of interpolation and quadrature. The remaining portion gives us probability (50 pp.) and elementary statistics (40 pp.). These subjects are treated in a simple way and illustrated by worked examples. The remaining pages are taken up with miscellaneous examples, answers and index.

As to the finite differences, it may be of interest to consider the place of this subject in any organised course of mathematics. If treated at all it is usually placed after the calculus and is so shown in the excellent little work on interpolation and approximate quadrature, by Dr. Gerhard Kowalewski (1932), who gives as a sub-title " An appendix to the text-books of differential and integral calculus ". But many useful formulae can be obtained for interpolation and quadrature by elementary algebra and presented in such a way that they may be understood and used by those whose knowledge of the calculus is very limited. Finite Differences as a whole form an extensive and difficult part of mathematics, as we are presented with a complicated arrangement of symbols. Our usual constants become for the time variables, and their place is taken by selected values of the variables; we have symbols of operation, symbols denoting structure and those representing quantities. For purposes of clear reasoning it is desirable that symbols of operation and of structure should be distinct, and they should never be used as quantives. A slight change of notation has been used in this work, namely $n^{(-r)}$ to represent $1/(n+1)(n+2)\ldots(n+r)$. Now, we can define the structural

symbol $n^{(r)}$ as standing for the quantive $n(n-1)\ldots(n-r+1)$; we might then agree to denote the reverse of this quantive, namely $n(n+1)\ldots(n+r+1)$ by $n^{(-r)}$, the inverses of the quantives being represented by $1/n^{(r)}$ and $1/n^{(-r)}$ respectively. This would require $1/(n+1)(n+2)\ldots(n+r)$ to be represented by $1/(n+1)^{(-r)}$ and not as in the present work. The reason for the change is the desire to be able to write $n^{(r)}(n-r)^{(m)}=n^{(r+m)}$. This is not necessary and is making the laws of thinking subservient to our symbolic convenience. This is a great danger to the student, who will be likely to work by rule instead of by doing a piece of clear thinking. The method by which the author derives $x^{(-m)}$ from $x^{(m)}$ by introducing a convention as required is not satisfactory and is a mixing of our different ways of thinking. De Morgan puts this clearly: when you say "every equation has a root" you are thinking of quantity; and when you say "every function has an inverse" you are thinking of structure. It is confusing to put m equal to 0 when you are thinking of the quantity, and make it "0" in the structure at the same time, for though 0 may have no value it may have consequences. The author obtains the desired result by a new convention.

A new symbol of operation has been introduced to represent divided differences, namely a Δ divided by a vertical line. It is a suggestive symbol, but it is nearly an axiom that symbols of operation should not be introduced without necessity. When we consider the five notations given on p. 40, all of them satisfactory for the purpose and none of them demanding any symbol that is not in constant use, it seems hardly desirable to add another which requires a number of subscripts to represent the arguments.

There is a short but clear account of osculatory interpolation and of Dr. Aitken's theorem on polynomial interpolation; this will be found of much interest, but one wishes that the author had chosen some other letters for his operators rather than the much-used Θ and θ.

The detail with which Mr. Freeman has worked his illustrative examples shows that he expects some of his readers to be at a very early stage in their mathematics. Such students cannot safely be trusted to use symbols of operation in their inverse form, as such symbols are interrogative and may have numerous answers.

The whole work is beautifully printed, contains much matter collected from various sources, and should be found of great practical value to all computers.

W. S.

Elementary Mechanics with Hydrostatics. By D. HUMPHREY and E. A. BAGGOTT. Pp. xvi, 628. 8s. Mechanics, 6s.; hydrostatics, 3s. 1939. (Longmans, Green)

It is not difficult to pick out details of this book for destructive criticism. In it we have, side by side, experimental treatment, results of observation of common appliances and everyday happenings, deductive logic, and a few *ex cathedra* statements. The careful but inexperienced reader may well sometimes be puzzled whether a result has been "proved" or obtained by experiments he has performed or whether it is stated on the basis of experiments or deductions not given in the text. Thus on p. 420, "Now it may be shown that for a simple pendulum $t=2\pi\sqrt{(l/g)}$". The reader is not told that this can be shown only by methods outside the scope of the book.

The critical portion of any book on elementary mechanics is the section on mass and weight; this is rarely quite satisfactory. P. 382 has a specious air of logic about it, but as it stands it simply will not do. The three stages are:

(i) "The results of the experiment show that there is a property of a body, *not its weight*, which is brought into play when we change the motion

of the body. It is this property which we call the *mass* of the body. The greater the effect of the property in the case of any body, the greater is the mass of the body. That there is a close connection between the mass of a body and its weight is evident. . . ."

(ii) "Bodies of equal weight when situated near each other fall freely with the same acceleration. . . . Hence bodies which have equal weights have equal masses."

(iii) "The weight-force provides a very convenient means of measuring mass, for if W_1 and W_2 are the weights of two bodies in the same units whose masses are m_1 and m_2, $W_1/W_2 = m_1/m_2$."

With regard to this, an intelligent reader will naturally raise the objection to (ii) that bodies of *unequal* weights fall freely with the same acceleration. Moreover, what is meant by "for if" in (iii)? Is the statement of proportion meant to be a deduction from (i) and (ii)?

The authors make a curiously tentative approach to calculus notation on p. 156: "The value of the ratio $\Delta s/\Delta t$ when Δt is made smaller and smaller is the instantaneous speed at the time t". The idea of limiting value is not explained, nor is the idea pursued, for example, in connection with acceleration.

But it is more pleasant and fairer to say that in general terms the book is of great interest and much originality. The authors have done some hard thinking on the order of presentation of the fundamental ideas of mechanics. As the basis is to be largely experimental, there is a preliminary chapter on measurement with some valuable remarks and exercises on degree of accuracy. Realising the weakness of an approach to statics which plunges the pupil straight into moments or the parallelogram of forces, the authors insert a Chapter II on "first experiments with forces" containing much common sense. It is admirable to find thus early the direction: "It is important when we describe a force that we should state: (*a*) The body on which it acts; (*b*) the direction in which it acts upon that body; (*c*) the magnitude of the force". This is intended to clear up any trouble about that mystic word "Reaction". It is surprising to find in the first paragraph of this chapter the correct definition of force—Newton's First Law, in fact—led up to by the results of observation. After all, the ordinary treatment of machines is by work, and the concept of work includes that of motion, so it is reasonable to speak early in the mechanics course of uniform motion, even if the term is incompletely defined at this stage. Chapters follow on parallel forces, centre of gravity, moments, machines (treated first by moments, then by work). Then we find three chapters on kinematics—speed, velocity, acceleration, falling bodies. Then, at last, the composition and resolution of forces and friction. Finally, the dynamics course in a usual order: laws of motion, projectiles, relative velocity, work and energy, momentum, impact.

There is very much to praise in all this. The experiments do not demand elaborate apparatus, though many would require considerable skill and care to get results worth having. The text is brightly written, with many historical notes (for example, on the fundamental experiments of Galileo, Newton, Joule). The authors are always ready to add point to a principle by an application from daily life. The frontispiece of the Forth Bridge suggests at once that statics deals with real things, not with abstract forces represented by straight lines, and the section on bridge construction is, within its limits, very well done. It is refreshing to find the following suggested for class discussion: "Suppose that a machine could be invented which was able to give a greater output of work than the input of work, what would be some of the consequences?" Many of the examples to be worked—nearly 1000 in the complete

book—are drawn from public examinations. The diagrams and pictures—over 400 in the mechanics part alone—are exceptionally clearly drawn and attractively set out. With the reservations expressed in the opening paragraphs of this review, the text has been written with care and lucidity.

The book is long for an elementary textbook. Not many school timetables would allow of it being worked in full with all the experiments and numerical examples. It might have been well to have starred certain sections for omission, if necessary. Teachers of elementary mechanics should consider this book, of which the merits outweigh the minor defects.

The hydrostatics chapters deal with pressure, density, pumps, Boyle's law, the principle of Archimedes, with a final chapter on hydrometers, submarines and balloons. This section of the book possesses the same good qualities as the rest. The outlook is eminently practical, the applications to everyday things are many, and the diagrams are admirably set out.

H. E. P.

Remarques sur la formation de la théorie abstraite des ensembles. I. Préhistoire. La création de Cantor. Pp. 107. 25 fr. **II. Dedekind. Les axiomatisations (Zermelo, Fraenkel, Von Neumann).** By J. CAVAILLÈS. Pp. 97-145. 12 fr. 1938. Actualités scientifiques et industrielles, 606, 607 ; le progrès de l'esprit, VII, VIII. (Hermann, Paris)

Méthode axiomatique et Formalisme. I. Le problème du fondement des mathématiques. Pp. 75. 20 fr. **II. Axiomatique et système formel.** Pp. 76-124. 18 fr. **III. La non-contradiction de l'arithmétique.** By J. CAVAILLÈS. Pp. 125-194. 18 fr. 1938. Actualités scientifiques et industrielles, 608, 609, 610 ; le progrès de l'esprit, IX, X, XI. (Hermann, Paris)

Logic of algebra. By P. DIENES. Pp. 76. 18 fr. 1938. Actualités scientifiques et industrielles, 614 ; logique et méthodologie, III. (Hermann, Paris)

These little tracts deal with subjects on which it is important that every student of mathematics should learn to form his own judgment, free from traditional prejudice.

Dr. Cavaillès gives, in vigorous style, historical accounts of the development of the theory of sets and of general problems of the foundations of mathematics. The accounts should be comprehensible to a comparative beginner, and the detailed references will be of value to anyone desiring to make a deeper study.

Dr. Dienes gives his personal " realist " conception of the notion of number without attempting any systematic criticism of other schools of thought.

The series should serve as a useful introduction to parts of mathematics which are still full of controversy.

L. C. Y.

Protons, Neutrons, Neutrinos. By JACQUES SOLOMON. Pp. xiii, 228. 100 fr. 1939. (Gauthier-Villars)

In the present extremely unsettled state of the theory of atomic nuclei, anyone attempting to write a book on the subject runs a grave risk of finding the major portion of his work quite out of date by the time it is available to the scientific public. Dr. Solomon has been rather unfortunate in just this way for his book is largely devoted to a description of the field theory of neutrinos in relation to nuclear forces, a theory now largely superseded by the much more attractive meson, or heavy electron, theory of these forces. Unfortunately, however, the meson theory has not rendered the neutrino completely superfluous. It is still necessary to invoke it as a practically unobservable carrier of momentum, spin and energy to account for the apparent failure of the conservation laws for these quantities which is a feature

of β-radioactivity. For this reason Dr. Solomon's book is still of more than historical value for he gives a very clear and thorough account of β-ray phenomena in the first two chapters, which occupy nearly half the book. Chapter III deals with the theory of nuclear forces, the first part being concerned with the now unfashionable neutrino theory of their origin and the second with a rather uncritical account of the " phenomenological " theory of nuclear interaction. The next chapter gives an account of the attempts made to understand the anomalous magnetic moments of neutron and proton in terms of virtual emission and absorption of electron-neutrino pairs and is no longer valuable in detail, though the central idea has been taken over in the meson theory of the moments. An interesting discussion of isomeric nuclei, containing numerous references and descriptions of the abundant experimental work, is the subject of Chapter V. After a formal chapter, now of only historical value, we find a brief mention of the meson theory in the last chapter, sandwiched in between highly speculative hypotheses concerning the nature of neutrinos.

One feels, after reading this book, that it is a pity that it was written a year too soon, for the exposition is up to the standard of the best French scientific books. Nevertheless, well over one half of the book is still of value to the serious student of nuclear physics, apart from the copious references to both experimental and theoretical work which are extremely useful.

H. S. W. M.

The Theory and Use of the Complex Variable. An introduction. By S. L. GREEN. Pp. viii, 136. 10s. 6d. 1939. (Pitman)

Complex Variable and Operational Calculus with Technical Applications. By N. W. McLACHLAN. Pp. xi, 355. 25s. 1939. (Cambridge University Press)

These two books go well together. The former is the type of book on complex numbers that we have awaited for some time. It surveys most of the ground that is usually dealt with in books on Higher Trigonometry—to wit, elementary operations with complex numbers, vector representation, De Moivre's theorem, exponential, logarithmic and circular functions of complex numbers. All this is done in 45 pages and yet nothing of importance is omitted except perhaps that in proving that the exponential function obeys the law of indices it is optimistic to assume that the student is familiar with the conditions under which infinite series can be multiplied.

The other and larger part of the book is devoted to the use of functions of a complex variable in analysis ; at every stage applications to other branches of mathematics or to physics and engineering are introduced so that the science student will feel this is a live subject for him. For instance in the chapter on differentiation and integration there are some admirable pages on the use of conjugate functions in obtaining equipotentials and stream-lines. In this chapter Cauchy's theorem is proved by use of Stokes' theorem, and then the Taylor and Laurent series are proved for rational functions only.

This chapter is followed by two on conformal transformation and then our appetite is whetted by two short chapters, one on potential problems in electrostatics and hydrodynamics and the other on alternating current problems, but they do not satisfy our thirst ; they do not go far enough to enable the student to read more difficult works. We hope that the author will either expand these chapters if this book attains a second edition, which it soon should do, or that he will write a sequel on the applications of the complex variable in physics and engineering.

McLachlan's book is a comprehensive treatise on Heaviside's operational

calculus from the standpoint of the complex variable. It is divided into three parts. Part I is an introduction of 100 pages on the analysis of functions of a complex variable, the first chapters of which cover much the same ground as is covered in the first half of Green's book, only the procedure is more rapid. The reader is referred to other books for a proof of Cauchy's theorem but the Taylor and Laurent expansions are established. In addition to this work there is a chapter on residues followed by one on the integration of functions along the two contours: (*a*) from $c-i\infty$ to $c+i\infty$, (*b*) a loop round the origin from $-\infty-i0$ to $-\infty+i0$, which are called the Br_1 and Br_2 contours respectively because they were first used by Bromwich in connection with Heaviside's work. From the proof that under certain circumstances those contours are equivalent the student might think that the integral round the part of Br_2 on the left of the imaginary axis was zero because the integrand approached zero at infinity; a few extra lines here would make the argument clearer.

There are then two chapters on the use of the Br_1 and Br_2 contours in establishing contour integrals for the Gamma, error, and Bessel functions. In so many books on complex variables the choice of a suitable contour for the definition of a function appears to be almost a fluke, whereas in nearly all cases it has been discovered by working back from some series and using either the Br_1 or the Br_2 contour.

In Part II the groundwork of the operational calculus is laid, p being treated rather as an inversion parameter than as an operator. The operational formulae to be used in Part III are established, the operational form for a function being defined by means of Mellin's Inversion theorem and Bromwich's integral along the Br_1 contour. The solutions of linear differential equations are then found in terms of Bromwich's integral.

In the last chapter of this part Heaviside's unit function $H(t)$ is defined, but it is not made clear enough that $\frac{1}{p^n}H(t)=0$ when $t<0$ though it is clearly stated that $\frac{1}{p^n}H(t)=\frac{t^n}{n!}$ for $t>0$.

Part III is probably the most important part of the book. In it the analysis developed in Part II is applied to a multitude of practical problems such as: oscillations in ordinary electrical circuits, radio and television receivers, condenser microphones and telegraph lines; artificial electric lines and wave filters; extraction of heat in a refrigerator and various problems in heat diffusion. The variety of problems tackled is sufficient evidence of the importance of the subject.

There are eleven appendices, in which various theoretical points too difficult to insert in the text are discussed. There is also a very useful bibliography of 222 books and papers, which is frequently referred to in the text, so that the reader should have no difficulty in studying further any application in which he is particularly interested.

Taken as a whole this book is a very notable achievement, for the author has succeeded in presenting a difficult subject in a way which makes it comprehensible to any practical man who has a fair knowledge of mathematics. It is the only book published in this country, apart from the Cambridge mathematical tract by Dr. Harold Jeffreys, which deals with use of the complex variable in operational calculus and it should be indispensable to engineers and physicists who wish to master the theory of their subjects, but they must be prepared, as Dr. McLachlan says, to put in a considerable number of hours in studying the subject and not expect to master this branch of mathematics during a week-end.

After reading the whole book, however, one cannot help feeling that the operational forms established in Part II are not used sufficiently in Part III, where in nearly every problem the solution is first written out in terms of a contour integral. Surely one of Heaviside's chief reasons for using the operator p was to solve problems without long integration, interpreting the meaning of a function of p by means of some suitable expansion in series. With his intuition for taking the right step he could have solved the practical problems tackled in this book with hardly a reference to integrals. The difficulty for the average man is that he has not Heaviside's knack of deciding which type of expansion is suitable or valid. The definition of an operational form by means of Mellin's inversion theorem and its interpretation by Bromwich's integrals enables us to formulate rules for valid expansions, such as Heaviside's partial function rule, which in this book receives only a bare mention; then, when these rules have been formulated and a table of operational forms made, we are in a position to solve problems without constantly returning to contour integrals. Part III would be improved if there were some indications of how and when the contour integrals can be omitted and a function of p used as an operator in a similar way to that in which the beginner solves linear differential equations by the use of the operator D. H. V. L.

The Beacon Arithmetic. I. Pts. 1 and 2. II. Pts. 1 and 2. III. Pts. 1 and 2. By C. M. Fleming. Pp. 128. 1s. 9d. each part. 1939.

Teachers' Manual. By C. M. Fleming. Pp. viii, 165. 3s. 6d. 1939.

Number Reader. By C. M. Fleming and E. H. Grassam. Pp. 96. 1s. 6d. 1939. (Ginn)

The publication of a new first Arithmetic Book is not an unusual event but we have here a series of class textbooks of exceptional interest to teachers of arithmetic whatever the age of their pupils.

Three aims are stated in the foreword. The first, and the one most fully realised, is that of giving sufficient and "approximately equal practice in all the items of the basic tables and in each step of the essential processes in computation, problem-solving and measurement". This is admirably achieved by analysing the processes into finely graded steps and giving ample sets of exercises with tests of accuracy and speed for each step before a new difficulty is introduced. The greatest care appears to have been taken to ensure that the items of the addition, subtraction, multiplication and division tables have equal practice, not in the conventional serial form of the table but in haphazard arrangement so that each item produces the correct response when met in isolation from its neighbours. All teachers have known children who can only give 7×9 correctly when they have already met 7×8 or 6×9 according to the form of table they have memorised. It is revolutionary to provide for the learning of separate items before a tabular form is introduced but, although this is recommended in the Teachers' Manual and no set table is printed in any of the books, teachers are free to give the children a "table" to learn or to use for reference if they so wish. Whether they do so or not these books certainly provide the means by which an accurate recall of any single item can be ensured. Since items are in general required singly this type of learning by direct association is of great importance and has in the past suffered some neglect. Question-answer cards (not seen by the reviewer) are also issued for oral work and practice games.

The second aim of the series is to secure that weaknesses are discovered and remedied before they have caused general failure and discouragement. Five tests are provided on the work of each chapter and so planned that

remedial practice can easily be prescribed. These tests can also be used by the children themselves to measure and record their accuracy and speed.

The third aim is one to make a strong appeal to teachers of older children. The author attempts to give an understanding of the "language of arithmetic" and "adequate practice in reading for exact meaning"; by these means the difficulties some children find in solving even the simplest problems will be largely overcome. The Number Reader is "planned to give children a reading experience which will introduce them to the language of arithmetic". It is an attractive book of rhymes and stories with amusing illustrations and, to ensure effective comprehension, questions and exercises quite unlike the conventional "sum". The textbooks themselves provide for a continuous increase in vocabulary, the various phrases which imply one of the four basic processes being deliberately introduced; *e.g.* "one way of telling us to subtract is to say 'What was left?' or 'How many were left?'". There is perhaps here a danger that the phrase itself may be taken as a merely mechanical stimulus to the correct procedure without any reference to the concrete situation in which the need arises. The Teachers' Manual offers ingenious suggestions for projects and activities which will give the phrase meaning and wise teachers will have their own methods of giving the children adequate number experience. Even if actual situations are denied to the children the stories of the boys and girls of Beacon Street with which new words or processes are sometimes introduced by the author will do much to make the arithmetic what it should be, a normal and intelligible activity. It must here be noted that the Manual, besides being a treasure house of suggestions for games and oral work, contains an excellent account of the scheme on which the books are based and shows how they may be adapted to various classes and grades.

The most welcome feature of this series, one for which teachers have waited long, is the completeness of the plan on which it has been constructed. Eight books have been planned in addition to the Number Reader, two for each year of the Junior School. It is recognised that before children can undertake calculations in which an unseen number must be held in mind and used (as when 17 is added to 26) they must have achieved complete mastery of the required fundamental combinations of pairs of numbers less than 10. There are 390 of these basic relations between numbers from 0 to 9; they are introduced to the children six or seven at a time in a carefully planned sequence and with ample practice for each group. Book I Part I provides for the 100 addition and the 100 subtraction facts; addition is carried only as far as the type $75+7$. A little simple work on shillings and pence is given but nothing else is allowed to encroach on the mastering of the addition and subtraction tables and their applications in "stories". In Bk. I Pt. II subtraction with "borrowing" is begun with careful preparation for the method of equal additions. The multiplication tables from 0 to 5 are then practised, *i.e.* the items from 0×0 to 5×9 or 9×5. It is curious that no reference is made to this reversal of order in multiplying and still more strange that while 0×0 is the first item to be memorised 1×1 is omitted until Bk. II Pt. I. Division facts within the same range are included in the first year. Simple exercises on shillings and pence and on feet and inches complete the first two parts. In the second year short multiplication and division are thoroughly practised so that all the difficulties of zeros, carrying figures and remainders are systematically overcome. Addition and subtraction are extended to cover more than two rows and columns and further work on length and money is undertaken. It will be seen that these first two years are dedicated to acquiring a sure knowledge of the essential steps of arithmetic.

What remains to be done in the later volumes is the extension of these processes to larger numbers and to measures of weight, time, capacity, etc. The stages of long multiplication and long division are admirably arranged in Bk. III Pt. I though probably for many children the larger numbers of Pt. II are unnecessarily difficult. The introduction of weight and capacity seems undesirably late in view of the easy numbers involved and the children's pleasure in the actual operations of measuring liquids and weighing. Multiplication and division by 14 and 16 are rightly postponed until this stage but easy addition of pounds and ounces and of quarts and pints could profitably be taken earlier.

Some preparation for fractional work is included in these first six parts but no doubt Bk. IV will be largely concerned with formal processes with fractions. Many teachers will consider such work more suited to the third year and would welcome the postponement of long division to the fourth year. This would agree with the findings of those investigators whose work has clearly been carefully considered by the author in the planning of this series.

The Board of Education has in recent years drawn the attention of teachers to the need for some form of geometrical teaching in Junior Schools. An attempt is made in these books to introduce ideas of shape, area, angle, etc., through objects interesting to the children but these sections are the least successful and the least sound.

Arithmetic books have been growing more attractive in appearance for some years but this series shows a decided advance in arrangement, illustrations and type. Their appeal to children should be immediate. To teachers who like arithmetic to be learnt individually or in groups they will be equally welcome for they teach the children in terms that they can understand and give them exercises well within their powers. It may be thought that progress is too slow at the beginning, but experienced teachers will agree that it is in the first two years of formal arithmetic that children so often grow discouraged or show signs of strain.

Although the books are not free from slips ($4 \div 0$ is an example occurring in Bk. II Pt. I!) they bear signs of the most patient application of modern research to arithmetic teaching and should meet the needs of those teachers who have for long been exercised about the difficulties of combining sound training in accuracy and speed of computation with the lively interest and intelligent purpose which are essential to enjoyable learning. E. M. W.

Calculus. By Frederick H. Miller. Pp. xiv, 418. 15s. net. 1939. (John Wiley and Son, New York ; Chapman and Hall)

This is a large book and one heavy to handle ; but if put on the table it lies flat wherever it is opened. The print is beautifully clear. There are answers to most of the examples, a full index, and a set of appendices. The last comprises : A. Formulas (*sic*) for reference in Algebra, Geometry (but mensuration only), Trigonometry, Plane and Solid Geometry. With great submission this appendix is too kind to the students for whom this book is intended : they ought to know every one of these " formulas " by heart. B. The inevitable Greek alphabet. C. Five sets of numerical tables. D. A table of integrals, one hundred in number and filling seven pages! It is clear that a very great deal of trouble has been taken with this work, both by the author and the publishers.

To give an idea of the scope of the book the following is a summary of its contents : I. Variables, Functions, and Limits. II. The Derivative Concept. III. The Processes of Differentiation. IV. Differential and Numerical Approximations. V. Maxima and Minima. VI. Tangents, Normals, and

Curvature. VII. Time-Rates and Motion. VIII. Analytical Evaluation of Limits. IX. Partial Derivatives. X. Curve Tracing. XI. The Integral Concept. XII. The Processes of Integration. XIII. Definite Integrals. XIV. Geometrical Applications of Integration. XV. Physical Applications of Integration. XVI. Multiple Integrals. XVII. Infinite Series. XVIII. Differential Equations.

In his Preface the author says that "a strong effort has been made to produce a book that fulfils the needs of students who wish to make use of Calculus in science or engineering, and also of those whose first interest is in pure mathematics". A good example of the author's approach to the subject is his treatment of integration. "First the definite integral is defined as the limit of a certain sum. Then the definite integral as a function of its upper limit is analytically shown to possess a derivative, from which the aspect of an integral as an integral or an anti-derivative arises." One reason for this point of view is that "some students, taught that a definite integral is equal to the limit of a sum because they are both represented by an area, have difficulty in realising that a definite integral can ever mean anything but an area". But, alas! the author defeats his own object in Art. 65, by employing Fig. 54 which shows an area from which he gets his definition of an integral. His very words are "as indicated in Fig. 54". This is very regrettable indeed. Furthermore, the reviewer considers that the deliberate throwing overboard of the long-suffering Duhamel is also regrettable.

Next (in the order of the Preface), "The derivative of a function of a single variable is treated analytically before the slope interpretation is discussed. This avoids the difficulty of a student's thinking that a derivative means nothing but a slope." To support this, proofs are given in Art. 6 for the fundamental theorems on limits; but it is admitted that the proofs are not intended to be entirely rigorous.

Thus the author's definitions of derivative and integral are respectively:

$$(1)\quad \lim_{x_2 \to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1};$$

$$(2)\quad \lim_{n\to\infty} \sum_{k=1}^{n} f(\bar{x}_k)(x_k - x_{k-1}), \text{ where } x_{k-1} \leqq \bar{x}_k \leqq x_k.$$

Once this way of looking at things has been grasped, the reviewer has no hesitation in saying that the development of processes and applications runs smoothly enough, while admirable features of the book are the large number (118) of beautifully clear figures, and the very large number ("more than 2300") of suitable exercises.

Yet a strong case can be made out against the possibility of doing what the author has tried to do in this book. In order to enable them to proceed quickly to applications, science and engineering students ought to be given the fundamental ideas in pre-digested form, so to speak, by consideration of "slope" and "area". The pure mathematician, on the other hand, must needs have a strong enough digestion gradually to assimilate the whole hard truth.

N. M. G.

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